BRIDGING COURSE

A Mathematics Primer Course to help bridge the gap between School and University

SOLUTIONS

BRIDGING COURSE

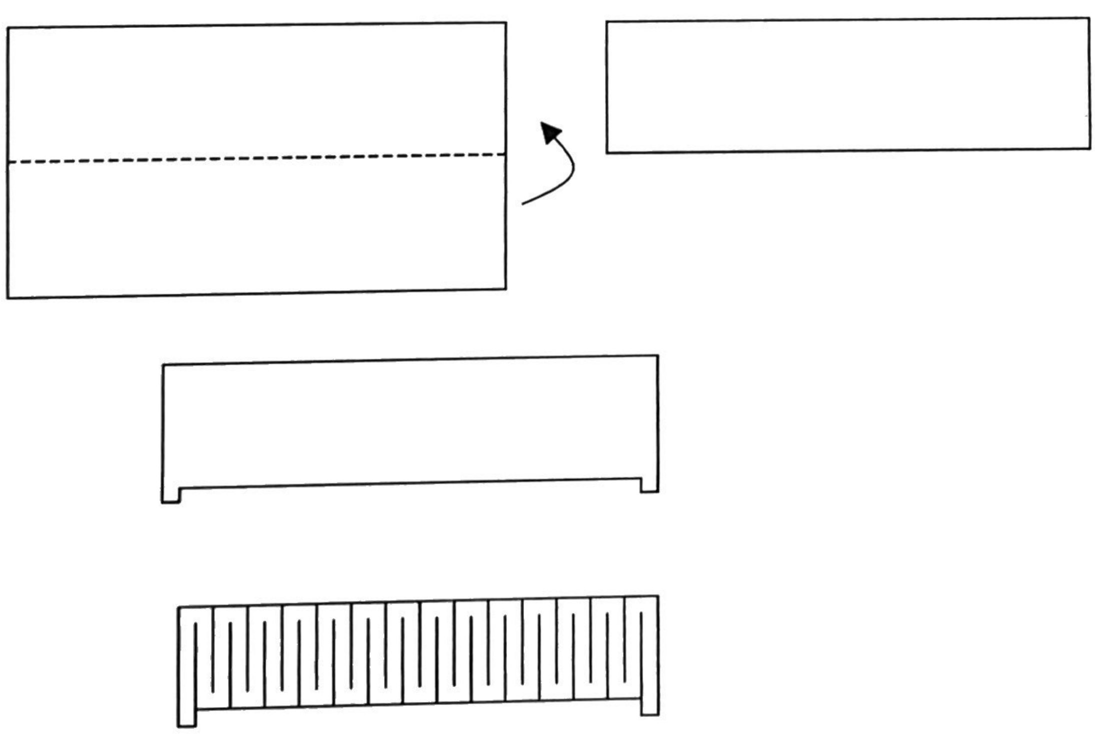
SOLUTIONS

Exercise 1.1 Solutions

Acknowledgement: I am not a puzzle expert, and admit that some of the puzzles have been adapted from a very old collection on my bookshelf entitled *The Children's Encyclopaedia*, but the title page, among others, is missing so I cannot give full credit. In turn the Encyclopaedia refers to a book by H.E.Dudeney called *Canterbury Puzzles* which I have not found but I am sure that there are lots of other books out there which provide equally stimulation challenges. In *More Mathematical Puzzles and Diversions*, Martin Gardner writes that Henry Ernest Dudeney was a great English inventor of puzzles who started publishing mathematical puzzles in magazines and newspapers from around 1890. *Canterbury Puzzles* was published in 1907 and is a mathematical version of Chaucer's Canterbury Tales. As here, in Dudeney's work, some of the puzzles are solved using simple geometry or mathematics while others require rather deeper mathematical explanations.

1. Try thinking of how you would make a long, continious ribbon from a sheet. Then think how you could add the hole add the hole, perhaps by folding.

   Take the sheet of A4 paper and fold it in half length ways. Then cut out the section shown at the fold. Make lots of vertical cuts as shown, each cut not quite reaching the edge of the remaining paper. Carefully unfold. Hey presto!



2. Put the spade in the ground standing upright. Lie on the floor and move your head towards, or away from the spade, until the top of the spade lines up with the top of the tree. Mark the place on the ground where your head is. Measure the distance to the spade, call it d, and the height of the spade which is above the ground, call it h. Measure the distance from the place where your head was to the base of the tree, D say. Let H be the height of the tree. Now, by the rule of similar triangles (or the definition of the tangent if you wish), $\frac{H}{D}=\frac{h}{d}$, which we can rearrange to find H, the height of the tree.

3. Let A be the number of apple trees, P be the number of pear trees, C be the number of cherry trees and T the total number of trees. We are told that one third of the total number of trees are apple trees, i.e. $\frac{1}{3}T=\frac{4}{12}T=A$, one quarter of the total are pear trees, i.e. $\frac{1}{4}T=\frac{3}{12}T=P$. Thus the apple and pear trees combined make $\frac{4}{12}T+\frac{3}{12}T=\frac{7}{12}T=A+P$. Thus the remaining trees must be $\frac{5}{12}$ of the total. That is $\frac{5}{12}T=C$ but $C=30$ and so $T=\frac{12}{5}\cdot 30=72$, thus there are 72 trees in total. There are other ways of course $\frac{1}{3}T=A \quad \Rightarrow \frac{A+P+C}{3}=A$ etc. but this is harder work.

4. Two children take the boat and one gets out at the land while the other returns with the boat. Next and adult goes across to the land where the child gets in and takes it back to the island. This process is then repeated for the other adult.

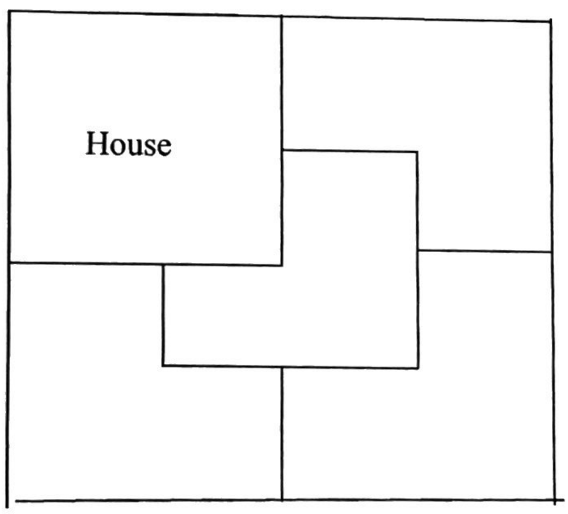
5. Row three is only 1 too many so think about knocking off the 2 and adding 1 at the beginning. Note also that the diagonals also add up to 34.

1	15	5	12
8	10	4	9
11	6	16	2
14	3	13	7

1	11	6	16
8	14	3	9
15	5	12	2
10	4	13	7

6. Achieved by three cuts. If we label the sides *a*, *b* and base *c*, then the first cut is from a point 1unit along *b* down from the apex to the midpoint of *c*. The second cut is from a point 3units along *b* from the apex meeting the first cut so that it makes an angle of 90 degrees. The final cut is midway along *a* so that it meets the first cut at an angle of 90 degrees. Then rotate the pieces so that the two portions of *c* are joined.

7.



8. Swap the 2 and the 7, two moves. Move the 4 to the centre the 9 to where the 4 was and the 5 to where the 9 was.

[2] [7][8] [1][5][6] [3][9] [4]

A total of only five moves.

9. Minimum 18

$$\begin{matrix} - & 9 & - \\ 9 & Q & 9 \\ - & 9 & - \end{matrix} = 36, \quad \begin{matrix} 1 & 7 & 1 \\ 7 & Q & 7 \\ 1 & 7 & 1 \end{matrix} = 32, \quad \begin{matrix} 1 & 8 & 0 \\ 8 & Q & 1 \\ 0 & 8 & 1 \end{matrix} = 34, \quad \begin{matrix} 1 & 7 & 1 \\ 7 & Q & 7 \\ 1 & 7 & 1 \end{matrix} = 32, \quad \begin{matrix} 2 & 5 & 2 \\ 5 & Q & 5 \\ 2 & 5 & 2 \end{matrix} = 28$$

$$\begin{matrix} 1 & 5 & 3 \\ 5 & Q & 5 \\ 3 & 5 & 1 \end{matrix} = 28, \quad \begin{matrix} 3 & 3 & 3 \\ 3 & Q & 3 \\ 3 & 3 & 1 \end{matrix} = 24, \quad \begin{matrix} 4 & 1 & 4 \\ 1 & Q & 1 \\ 4 & 1 & 4 \end{matrix} = 20$$

$$\begin{matrix} 5 & - & 4 \\ - & Q & - \\ 4 & - & 5 \end{matrix} = 18, \quad \begin{matrix} 6 & - & 3 \\ - & Q & - \\ 3 & - & 6 \end{matrix} = 18, \quad \begin{matrix} 8 & - & 1 \\ - & Q & - \\ 1 & - & 8 \end{matrix} = 18, \quad \begin{matrix} 7 & - & 2 \\ - & Q & - \\ 2 & - & 2 \end{matrix} = 18, \quad \begin{matrix} 9 & - & 0 \\ - & Q & - \\ 0 & - & 9 \end{matrix} = 18$$

10. We are given

$$\frac{\sum_{n=1}^{4} d_n}{4} = 18, \quad \frac{\sum_{n=1}^{7} d_n}{7} = 15 \Rightarrow \sum_{n=1}^{4} d_n = 72 \text{ and } \sum_{n=1}^{7} d_n = 105$$

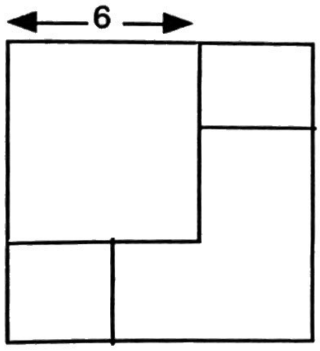
but

$$\sum_{n=1}^{7} d_n = \sum_{n=1}^{4} d_n + \sum_{n=5}^{7} d_n \Rightarrow \sum_{n=5}^{7} d_n = \sum_{n=1}^{7} d_n - \sum_{n=1}^{4} d_n = 105 - 72 = 33$$

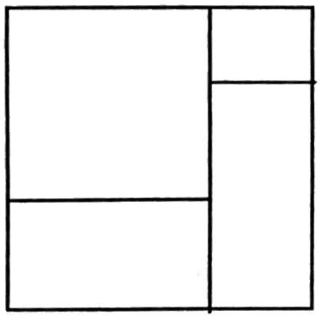
To find the average for the last 3 days

$$\frac{\sum_{n=1}^{4} d_n}{3} = \frac{33}{3} = 11.$$

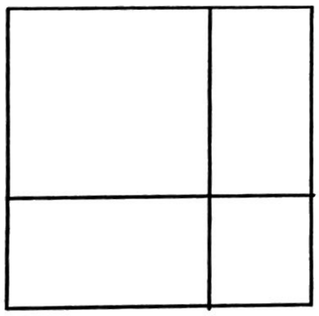
11.



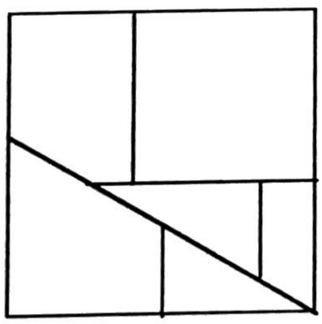
4 cuts and 2 sewn edges



3 cuts and 1 sewn edges

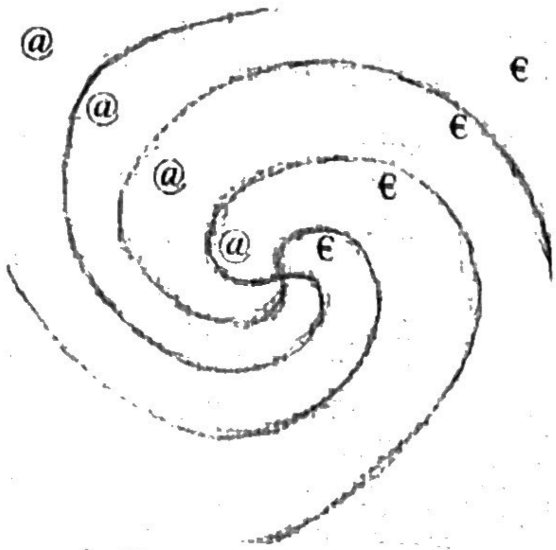


2 cuts and 1 sewn edges



5 cuts and 3 sewn edges

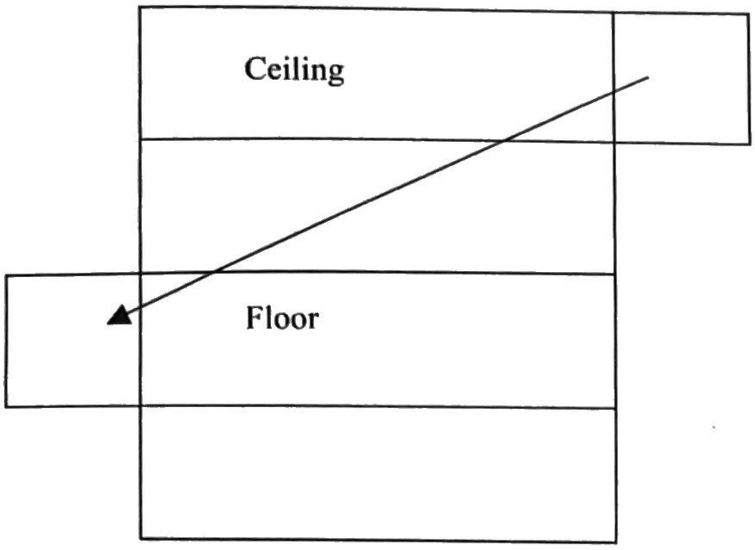
12. Tricky to draw but basically a spiral pattern



What is the equation for a spiral?

13. Take a sweet from the jar labled Dolly Mixtures. It is not the Dolly Mixtures, since the labels are wrong. Either it is a Sherbert Lemon or a Wine Gum. If it is a Sherbert Lemon then the jar labelled Wine Gums is really full of Dolly Mixtures while the one labelled Sherbert Lemons must have the Wine Gums. Alternatively, if the sweet selected was a Wine Gum, then the jar labelled Sherbert Lemons is really full of Dolly Mixtures while the one labelled Wine Gums must have the Sherbert Lemons.

14. Basically cut the room and lay it out flat.



We can form a triangle with the straight line from the spider to the fly as the hypotenuse, h. The base is 32 and the height 24. Therefore using Pythagarus's theorem

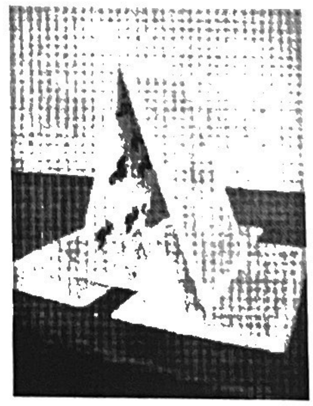
$$h^2 = 32^2 + 24^2 = 4^2(8^2 + 6^2) = 4^2 \cdot 10^2 = 40$$

and thus we see that the spider has to travel 40 ft. This is shorter than going straight down the end wall (11 ft) across the floor (30 ft) and then up 1 ft, giving a total 42 ft. There ought to be a metric version really but as this was taken from an old book (see comment at the end of the exercise) I prefer to keep the old measures. It makes you wonder though, when will we go entirely metric?

15. We have been given $M + P = 100$, $E + M = 90$, $P + E = 70$. Solving these gives $E = 70 - P$. Hence $M = 20 + P$ and $P = 40$, $M = 60$ and $E = 30$.

16. Cut out the holes only and slide the remaining piece in. Tricky to draw – have a go.

17. Lay 3 cards on the table, 2 side by side with the third on top going across the two. The remaining two stand up leaning against each other.



18. There are two similar ways. We describe one. The train driver goes forward and then backs up the line to Sidcup to hitch up with the carriage B. He then continues back and pushes it under the bridge before uncoupling. He then goes forward and reverses then goes up the side track towards Dartford to connect carriage A to the front. He then moves forward a little more so that he can connect carriage A to carriage B. He then reverses past the points and moves forward to leave carriage B in the centre where the engine was at the start. He then goes back, then forward towards Dartford again to push carriage A under the tunnel. He uncouples and then goes back to pick up B which, by first reversing, he can now take and leave at Dartford. He then goes down, back along the straight track, reverses up past Sidcup to pick up carriage A leaving it at Sidcup before going on his way. Phew!

19. It takes two and a half hours. The key point to bring to this problem is the fact that

$$\text{flow rate} = \frac{\text{capacity}}{\text{time}}$$

Let F be the tap flow rate, f be the flow rate out of the hole and V be the volume or capacity. We are given that

$$F = \frac{V}{2}, \quad f = \frac{V}{10}$$

When both are acting the hole empties the water so we need a negative sign,

$$Time = \frac{V}{F-f} = \frac{V}{V/2 - V/10} = \frac{10}{4}$$

20. 291

21. One (but unlikely) possibility is the North Pole. Another other option might be 1km North of the South Pole. This has the problem that if I travel 1km to the South Pole then when it comes to travelling 1km East I have to march on the spot. Do I turn round? If so I do not get back to where I started. Or, just above the South Pole is a latitude (α) whose circumference is exactly 1km. If I lived anywhere on a latitude (β) 1km above α then I will be able to travel 1km to α, 1km East will take me completely around the circumference of the special latitude at α so that when I travel 1km North I end up back at home. But I can be anywhere on the latitude β.

22. If we denote the large sheet as having length $2a$ then the cut out squares must have length $a/3$.

23. I do not know a fancy way to find single digit integers a, b and c such that $a^{(b+c)} = bca$. Clearly $a = 1$ does not work so it is a matter of going through the numbers with brute force. Perhaps the best way to do this is to take each number in turn and consider its powers which consist of 3 digits. If $a = 2$ then the only power that gives a 3 digit number that ends in 2 is $2^9 = 512$, so this does not work. For $a = 3$ the only power that ends in 3 is $3^5 = 243$ but this does not work since $2 + 4 = 6$ and not 5. Nothing works for $a = 4$, but $5^3 = 5^{1+2} = 125$ so we have found an answer. Furthermore $6^3 = 216 = 6^{2+1}$ is another.

24. Again I do not know a fancy way. Clearly all multiples of 10 work, such as 120 is divisible by 12. Also the numbers 22, 33, ... All work. Are there others?

25. The chances that the box selected on the first attempt contains the key is 1/3. However on the second choice the chances are now 1/2 so theretically they should always swap. However this highlights the issue that this may be statistically the best bet but on a once in a lifetime chance would you do it?

26. There is a long but straight forward solution which we give the start here. First number the coins and split into two groups of 4. TRIAL 1: put 1, 2, 3, 4 against 5, 6, 7, 8. If the scale balances then the counterfeit is in 9,10,11,12. If if does not balance then you know that 9,10,11,12 are all genuine. In which case TRIAL 2: take 3 genuine coins, say 9,10,11 and put them against 1, 2, 3. Can you work it out from here?

Exercise 2.1 Solutions

1. I) Even + Even = Even

n, m both even $\Rightarrow$ there exists integers k and p such that $n = 2k$ and $m = 2p$. Then $n + m = 2k + 2p = 2(k + p)$. Since k and p are integers then $k + p$ is an integer and this implies that $n + m = 2 \times (\text{integer})$ which by definition means that $n + m$ is an even number.

II) Odd + Odd = Even

n, m both odd $\Rightarrow$ there exists integers k and p such that $n = 2k + 1$ and $m = 2p + 1$. Then $n + m = 2k + 1 + 2p + 1 = 2(k + p + 1)$. Since k and p are integers then $k + p + 1$ is an integer and this implies that $n + m = 2\,(\text{integer})$ which by definition means that $n + m$ is an even number.

III) Even + Odd = Odd

n even and m odd $\Rightarrow$ there exists integers k and p such that $n = 2k$ and $m = 2p + 1$. Then $n + m = 2k + 2p + 1 = 2(k + p) + 1$. Since k and p are integers then $k + p$ is an integer and this implies that $n + m = 2(\text{integer}) + 1$ which by definition means that $n + m$ is an odd number.

IV) Even × Even = Even

n, m both even $\Rightarrow$ there exists integers k and p such that $n = 2k$ and $m = 2p$. Then $n \times m = 2k \times 2p = 2(2k \times p)$. Since k and p are integers then $2 \times k \times p$ is an integer and this implies that $n \times m = 2(\text{integer})$ which by definition means that $n + m$ is an even number.

V) Odd $\times$ Odd $=$ Odd

n, m both odd $\Rightarrow$ there exists integers k and p such that $n = 2k+1$ and $m = 2p+1$. Then $n \times m = (2k+1)\times(2p+1) = 2(2kp + p + k)+1$. Since k and p are integers then $2kp + k + p$ is an integer and this implies that $n \times m = 2(\text{integer})+1$ which by definition means that $n+m$ is an odd number.

VI) Even $\times$ Odd $=$ Even

n even m odd $\Rightarrow$ there exists integers k and p such that $n = 2k$ and $m = 2p+1$. Then $n \times m = (2k)\times(2p+1) = 2(2kp + k)$. Since k and p are integers then $2kp + k$ is an integer and this implies that $n \times m = 2(\text{integer})$ which by definition means that $n+m$ is an even number.

2. Show

Exercise 2.2 Solutions

1. $\left(\frac{27}{8}\right)^{-2/3} = \frac{4}{9}$

2. i) $\dfrac{\sqrt{(1+x)} - \dfrac{x}{2\sqrt{(x+1)}}}{1+x} = (1+x)^{-3/2}\left\{\dfrac{2+x}{2}\right\}$

   ii) $\dfrac{(1-x)^{1/2}}{2\sqrt{1+x}} + \dfrac{(1+x)^{1/2}}{2\sqrt{1-x}} = (1-x^2)^{-1/2}$

3. i) $x^2 \div x^{-2} = x^4$

   ii) $a^3b^4ca^4 \div ab^2c^3 = a^6b^2c^{-2}$

   iii) $\dfrac{10^7}{10^9} = 10^{-2}$

   iv) $x/(y/z) = \dfrac{xz}{y}$

   v) $(x/y)/z = \dfrac{x}{yz}$

   vi) $\sqrt{a} \times a\sqrt{b} \times \sqrt{c}/(ab^{1/2}c) = \sqrt{\dfrac{a}{c}}$

   vii) $e^{(a+b)}e^{-b}e^{b-2a} = e^{b-a}$

   viii) $\dfrac{2^5 \times 3^2 \times 5^{5/2}}{2^3 \times 3^{-2} \times 5^2} = 324\sqrt{5}$

   ix) $x^{1/2} \div \sqrt[3]{x} = x^{1/6}$

   x) $\sqrt{x}/\sqrt[3]{x} = x^{1/6}$

4. i) $8^n \times 2^{2n} \div 4^{3n} = 2^{-n}$

   ii) $3^{n+1} \times 9^n \div 27^{2n/3} = 3^{n+1}$

   iii) $\dfrac{x^{p+q/2} \cdot y^{2p-q}}{x^p \cdot y^{2p} \cdot x^{q/2}} = y^{-q}$

5. $222 \approx 2 \times 10^2$ $\quad 22^2 \approx 5 \times 10^2$ $\quad 2^{22} \approx 4 \times 10^6$ $\quad 2^{2^2} \approx 10^1$

   $2222 \approx 2 \times 10^3$ $\quad 222^2 \approx 5 \times 10^4$ $\quad 22^{22} \approx 3 \times 10^{29}$ $\quad 22^{2^2} \approx 2 \times 10^5$

   $2^{22^2} = 2^{484} = \left(2^{242}\right)^2 = \left(7 \times 10^{72}\right)^2 \approx 5 \times 10^{145}$ $\quad 2^{2^{2^2}} \approx 6 \times 10^4$ $\quad 2^{2^{22}} \approx 10^{1,265,236}$

6. $x^6 - 54x^4 - 4x^3 + 972x^2 - 216x - 5828 = 0$

7. Hertz – one cyle per second, GHz stands for giga Hertz $= 10^9$ cycles per second.

8. Nano means 10^{-9} metres

Exercise 2.3 Solutions

1. Use the method of contradiction. We assume that rational solutions exist. That is there exists integers m, n, p and q such that we can write

$$ax^2 + bx + c = a\left(x - \frac{m}{n}\right)\left(x - \frac{p}{q}\right)$$

Importantly we assume that m, n, p and q have no common factors, otherwise we would just cancel them down. Multiplying out the bracket and then comaring coefficients of x we can see that m, n, p and q must satisfy

$$bnq = -a(mq + pn)$$

and

$$cnq = amp$$

We are given that a, b and c are odd integers. We now just use the results from Exercise 2.1 to show that this assumption of rational solutions leads to a contradiction. For instance consider

$$bnq = -a(mq + pn)$$

then to start assume that m is even. This means that n cannot be even, since m and n are co-prime and so n is odd.

Now, mq is even since m is even. Let us further assume that p is even this implies that $-a(mq + pn)$ is even since we know an Even + Even = Even. This tells us that the left hand side, bnq, must be even and so q must be even. But this means both p and q are even contadicts our assumption that these are co-prime. So it tells us that our last assumption that p is even must be wrong.

You can continue in this manner to rule out all possible options, in which case the only possible outcome is that our original assumption must be wrong and so no rational solutions exist.

2. Show

Exercise 2.4 Solutions

1. $|x-5|=8 \quad \Rightarrow x=13,-3$

2. $|x-1|=|x+3| \quad \Rightarrow x=-1$

3. $1<|x+2|<3 \quad \Rightarrow -1<x<1$ and $-5<x<-3$

4. $|2x-1|=|x+2| \quad \Rightarrow x=3,-1/3$

5. $|x-3|<|x+1| \quad \Rightarrow x>1$

6. $2x \le |x+3| \quad \Rightarrow x<3$

7. $|x^2-5|>1 \quad \Rightarrow -2<x<2$ and $-\sqrt{6}>x>\sqrt{6}$

8. $|x^2-4| \le 5 \quad \Rightarrow -3 \le x \le 3$

9. $\left|\dfrac{3x+4}{x-6}\right|<1 \quad \Rightarrow -5<x<1/2$

10. $\left|\dfrac{5x+9}{3x-1}\right|<1 \quad \Rightarrow -5<x<-1$

Exercise 2.5 Solutions

1. i) $4x+7>-5 \Rightarrow x>-3$

   ii) $17-3x>2 \Rightarrow x<5$

2. i) $1<2x-9<11 \Rightarrow 5<x<10$

   ii) $\frac{9x+1}{10}<\frac{9x}{10}+1$, true for all x

3. i) $\frac{9x+1}{10}>\frac{9x}{10}+1$, never true

   ii) $-5\le\frac{20-x}{4}\le 5 \Rightarrow 40\ge x\ge 0$

4. i) $\frac{1}{2}(x+5)-5>\frac{x}{3} \Rightarrow x>15$

   ii) $x^2+2>3x \Rightarrow$ either $x<1$ or $x>2$

5. i) $x^2+3x+2>0 \Rightarrow$ either $x<-2$ or $x>-1$

   ii) $x^2+8<2x$ not satisfied by any real valued x

6. i) $x(x+2)<3 \Rightarrow -3<x<1$

   ii) $x-4>\frac{-4}{x} \Rightarrow x>0$

7. i) $x>\frac{1}{x} \Rightarrow x>1$ and $-1<x<0$

   ii) $x-\frac{20}{x+1}<0 \Rightarrow -1<x<4$ *or* $x<-5$

8. i) $x>x^3 \Rightarrow x<-1$ or $0<x<1$

   ii) $x^3-5x^2+6x<0 \Rightarrow 2<x<3$ and $x<0$

9. i) $-2 < x^2 + 3x < 10 \Rightarrow -1 < x < 2$

ii) $\dfrac{x+3}{1-2x} > 1 \Rightarrow -2/3 < x < 1/2$

10. $\dfrac{2x^2 - 3x + 1}{x^2 - 1} > 1 \Rightarrow x < -1$ or $x > 2$

Exercise 2.6 Solutions

1. Consider b and $d > 0$ or b and $d < 0$. Then consider what happens if b and d are different signs. Are there any other restrictions needed?

2. Try $a = 3, b = 4, c = 1$ and $d = 2$. Then $a < b$ and $c < d$ but $\frac{a}{c} = 3$.

3. First

$$(a-b)^2 \geq 0 \quad \Rightarrow a^2 - 2ab + b^2 \geq 0$$

and

$$\Rightarrow (a+b)^2 - 4ab \geq 0 \quad \Rightarrow a + b \geq 2\sqrt{ab}$$

Note $\frac{a+b}{2} \geq \sqrt{ab}$ means that the *arithmetic mean* $\geq$ *geometric mean*.

This also tells us that $a^2 + b^2 \geq 2ab \quad \Rightarrow a^2 + b^2 \geq 2|ab|$

4. Consider left hand side and right hand side separately.

$$cb < ad \quad \Rightarrow cb + cmd < ad + cmd$$
$$\Rightarrow c(b + md) < d(a + cm)$$

Hence LHS result. Same for RHS

5. To show that $|a+b| \leq |a| + |b|$ use the definition $-|x| \leq x \leq |x|$.

6. Consider $(a+b)^2 + (c+d)^2 \geq 0$ and then expand.

Exercise 3.1 Solutions

1. i) $(2x+y)^3 = 8x^3 + 12x^2y + 6xy^2 + y^3$

ii) $(3x-y)^4 = 81x^4 - 108x^3y + 54x^2y^2 - 12xy^3 + y^4$

iii) $\left(2x+\frac{1}{x}\right)^6 = 64x^6 + 192x^4 + 240x^2 + 160 + \frac{60}{x^2} + \frac{12}{x^4} + \frac{1}{x^6}$

2. $\frac{1}{2}\times\frac{3}{4}\times\frac{5}{6}\times \dots \times\frac{2n-1}{2n} = \frac{(2n-1)!}{2^{2n-1}n!(n-1)!}$

3. i) ${}^nC_0 = 1$

ii) ${}^nC_1 = n$

iii) ${}^nC_n = 1$

iv) ${}^4C_2 = 6$

v) $\binom{5}{2} = 10$

vi) $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$

4. i), ii) and iii) start with one side and show that it equals the other side.

iv) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ is the formula for calculating the binomial coefficients using Pascal's triangle.

5. Just add them up and see it agrees but this observation does not constitute a proof at this stage. In Exercise 4.5, Number 8 you will be asked to prove this result.

6. i) Called Vandermonde's theorem. Expand $(1+x)^{m+n}$ and $(1+x)^m(1+x)^n$, assuming that we can multiply two infinite expressions, and compare the coefficient of x^k

$$\binom{m+n}{k} = \sum_{j=0}^{k}\binom{m}{k-j}\binom{n}{j}$$

ii) Rewriting we have $\left({}^nC_0\right)^2+\left({}^nC_1\right)^2+\left({}^nC_2\right)^2+\ldots\ldots+\left({}^nC_n\right)^2=\sum_{j=0}^{n}\binom{n}{j}^2$

Looking at the formula above in part (i) if we let $m=n$ and $k=n$ then this gives $\binom{2n}{n}=\sum_{j=0}^{k}\binom{n}{n-j}\binom{n}{j}$

which, using the result of Question 2(i) we have

$$\left({}^nC_0\right)^2+\left({}^nC_1\right)^2+\left({}^nC_2\right)^2+\ldots\ldots+\left({}^nC_n\right)^2=\binom{2n}{n}=\frac{(2n)!}{((n)!)^2}$$

7. Using integration by parts $\int_0^\infty x^n e^{-x}dx=n!$, compare with the Gamma function defined as $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$, note that the t is a dummy variable. Then when x is a positive integer $\Gamma(n)=\int_0^\infty t^{n-1}e^{-t}dt=(n-1)!$

8. If there are n objects to arrange but p of them are alike then the number of ways is given by $N=\frac{n!}{p!}$. If there are q objects, which are alike, but these are different to p then the number is $N=\frac{n!}{p!q!}$.

   i) In this case $n=4$ and $p=2$ therefore $N=\frac{4!}{2!}=12$.

   ii) There are 5 letters but 3 d's so $N=\frac{5!}{3!}=20$

   iii) There are 6 letters but 2 o's and 2 n's so $N=\frac{6!}{2!2!}=180$

9. The number of permutations is 8! But if the table is circular then it does not matter which particular seat I am in, only who is to my left and right. This means that if I fix myself somewhere this reduces the number of permutations to 7!=5040

10. Just use the binomial expansion.

11. This should have been covered in the notes. You do it by expanding using the binomial theorem. The answer is Yes. Consider the fraction $\frac{101^n - 99^n}{100^n}$ and check to see if it is greater or less than 1.

Exercise 3.2 Solutions

1. i) $x^2 - 16 = (x-4)(x+4)$

   ii) $25x^2 - y^2 = (5x-y)(5x+y)$

   iii) $9a^2 - 4(b-c)^2 = (3a - 2[b-c])(3a + 2[b-c])$

   iv) $z^4 - 1 = (z^2 - 1)(z^2 + 1)$

   v) $108 - 3z^2 = 3(6-z)(6+z)$

   vi) $4(a-b)^2 - 25(c-d)^2 = (2[a-b] - 5[c-d])(2[a-b] + 5[c-d])$

2. i) $x^2 + 9x + 20 = (x+5)(x+4)$

   ii) $x^2 - 9x + 20 = (x-5)(x-4)$

   iii) $x^2 - x - 20 = (x-5)(x+4)$

   iv) $x^2 + x - 20 = (x+5)(x-4)$

   v) $x^2 + 5x + 6 = (x+2)(x+3)$

   vi) $x^2 - 8x + 15 = (x-5)(x-3)$

   vii) $x^2 - x - 6 = (x-3)(x+2)$

   viii) $x^2 + 11x + 18 = (x+2)(x+9)$

   ix) $x^2 + x - 2 = (x+2)(x-1)$

   x) $12 + z - z^2 = (4-z)(3+z)$

3. i) $12x^2 + 7x - 10 = (3x-2)(4x+5)$

   ii) $2x^2 + 11x + 15 = (2x+5)(x+3)$

   iii) $7y^2 - 19y - 6 = (7y+2)(y-3)$

   iv) $3 + x - 2x^2 = (3-2x)(x+1)$

   v) $12y^2 + 11y + 2 = (4y+1)(3y+2)$

   vi) $6l^2 - 17lm + 12m^2 = (3l-4m)(2l-3m)$

   vii) $15x^2 - 17x - 4 = (3x-4)(5x+1)$

   viii) $2z^2 - 2z - 1 = (2z+1)(z-1)$

4. i) $px + pq - 6x - 6q = (p-6)(x+q)$

ii) $x + y - ax - ay = (1-a)(x+y)$

iii) $x^2 - y^2 - 6x + 6y = (x+y-6)(x-y)$

iv) $x^2 - (y-5)x - 5y = (x+5)(x-y)$

v) $1 - a^2 - 2ab - b^2 = (1-a-b)(1+a+b)$

vi) $a^2 + 5a + 6 + ax + 2x = (a+2)(a+3+x)$

vii) $4x^2 - 4xy + 5x + 1 - y = (4x+1)(x-y+1)$

5. i) $x^2 = 1 \Rightarrow (x-1)(x+1) = 0 \Rightarrow x = \pm 1$

ii) $x^2 - 3x + 2 = 0 \Rightarrow (x-1)(x-2) = 0 \Rightarrow x = 1, 2$

iii) $3x^2 - 17x = -10 \Rightarrow (3x-2)(x-5) = 0 \Rightarrow x = 5, 2/3$

iv) $5x^2 - 7x = 0 \Rightarrow x(5x-7) = 0 \Rightarrow x = 0, 7/5$

v) $x^2 + 4x + 4 = 0 \Rightarrow (x+2)^2 = 0 \Rightarrow x = -2, -2$

vi) $x^2 + 10x + 9 = 0 \Rightarrow (x+1)(x+9) = 0 \Rightarrow x = -1, -9$

vii) $y^2 - y - 42 = 0 \Rightarrow (y-7)(y+6) = 0 \Rightarrow y = 7, -6$

6. i) $x^3 + 8y^3 = (x+2y)(x^2 - 2xy + 4y^2)$

ii) $8 - 27b^3 = (2-3b)(4+6b+9b^2)$

iii) $z^6 + 1 = (z^2+1)(z^4 - z^2 + 1)$

iv) $ab^3 - 8a = a(b-2)(b^2 + 2b + 4)$

7. $100^2 - 99^2 = (100-99)(100+99) = 199$

8. Let $z = (x^2 + 2x + 1)(x^2 - 6x + 9) = (x+1)^2(x-3)^2$, hence $\Rightarrow \sqrt{z} = (x+1)(x-3)$

9. $(x^2 + 6x) = (x+3)^2 - 9$ i.e. add 9

10. $(x+y)^2 - (xy+1)^2 = (x-1)(x+1)(1-y)(1+y)$

11. i) $a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$

ii) $a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$

iii) $a^2 + b^2 + c^2 + 2(ab + bc + ac) = (a+b+c)^2$

iv) $a^3 + b^3 + c^3 - 3abc = (a+b+c)\left(a^2 + b^2 + c^2 - ab = bc - ac\right)$

12. $y = 1 - \dfrac{7}{36}x^2 + ...$

13. $a^n \pm b^n = (a \pm b)\left(a^{n-1} \mp a^{n-2}b + ... \mp ab^{n-2} + b^{n-1}\right)$

14. All of these can be done by assuming that if n even the there exists an integer k such that $n = 2k$, while if m is odd then there exists an integer p such that $m = 2p + 1$.

15. Use the method of prrof by contradiction. Assume that rational roots exist and hence we can write $ax^2 + bx + c = a\left(x - \dfrac{m}{n}\right)\left(x - \dfrac{p}{q}\right)$ then use the results from Question 14 to prove that this assumption leads to a contradiction.

Exercise 3.3 Solutions

1. i) $x\left(x^3+3\right)=x^4+3x$

ii) $\sqrt{x}\left(\frac{1}{\sqrt{x}}+x+1\right)=1+x^{3/2}+x^{1/2}$

iii) $x\left(x-\frac{2}{x}+\frac{1}{x^2}\right)=x^2-2+\frac{1}{x}$

2. i) $\frac{1}{x}+\frac{2}{y}=\frac{y+2x}{xy}$

ii) $\frac{1}{x-1}+\frac{2}{x-5}=\frac{3x-7}{(x-1)(x-5)}$

iii) $\frac{1}{x}+\frac{4}{x^2}=\frac{x+4}{x^2}$

iv) $\frac{1}{1-x}+\frac{2x}{(x-1)^2}=\frac{x+1}{(x-1)^2}$

v) $\frac{1}{y}-\frac{1}{xy}=\frac{x-1}{xy}$

3. i) $3(x-a)-4(a-2x)+2a=11x-5a$

ii) $2a(b-a)-3b(2a+b)+4b=-2a^2-3b^2$

iii) $(a+2b)^2-(a-2b)(b+a)=b(5a+6b)$

iv) $(a+b+c)^3-(a^3+b^3+c^3)=3(a+b)(b+c)(c+a)$

v) $\frac{x^2-16}{x+4}=x-4$

4. i) $(x-y)(x+y)=x^2-y^2$

ii) $(x-y)^2=x^2-2xy+y^2$

iii) $\left(\sqrt{5}-1\right)\left(\sqrt{5}+1\right)=5-1=4$

iv) $\left(\sqrt{5}-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)=5-3=2$

v) $\left(2\sqrt{2}-1\right)\left(2\sqrt{2}+1\right)=8-1=7$

vi) $\left(2\sqrt{3}-3\sqrt{2}\right)\left(2\sqrt{3}+3\sqrt{2}\right)=12-18=-6$

vii) $\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right)=x-y$

viii) $\left(x^2+1\right)(x-1)=x^3-x^2+x-1$

ix) $(x+1)(x+2)=x^2+3x+2$

x) $\left(\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a-b}\right)=\frac{-1}{ab}$

5. i) $a(x-y)+b(x-y)=(a+b)(x-y)$

ii) $\frac{3}{x+1}+\frac{5}{x+1}=\frac{8}{x+1}$

iii) $\frac{7}{3(x-4)}+\frac{1}{x-4}=\frac{4}{3(x-4)}$

iv) $\frac{7}{3(x-4)}+\frac{2}{3(x+2)}=\frac{3x+2}{(x-4)(x+2)}$

v) $\frac{1}{x+1}-\frac{2}{2x+5}=\frac{3}{(x+1)(2x+5)}$

6. i) $\frac{1}{\sqrt{2}-1}=\sqrt{2}+1$

ii) $\frac{1}{\sqrt{2}+1}=\sqrt{2}-1$

iii) $\frac{1}{1-\sqrt{2}}=-1-\sqrt{2}$

iv) $\frac{3\sqrt{5}-2\sqrt{3}}{\sqrt{5}+\sqrt{3}}=\frac{21-5\sqrt{15}}{2}$

v) $\frac{\sqrt{3}+1}{\sqrt{3}-1}=2+\sqrt{3}$

vi) $\frac{x\sqrt{y}-\sqrt{x}}{\sqrt{x}-\sqrt{y}}=\frac{xy-x+(x-1)\sqrt{xy}}{x-y}$

7. Find $(a+b+c)^3$

Exercise 4.1 Solutions

1. $\sum_{k=1}^{n} 1 = 1 + 1 + 1 + ... + 1$ n times so the answer is n

2. These all feature in A Note Book in Pure Mathematics by L.H.Clarke but they are standard sums, indeed all were taught to me by Paddy Murphy.

   i) $1 + 2 + 3 + ... \quad + n = \dfrac{n(n+1)}{2}$

   ii) $1.2 + 2.3 + 3.4 + ... \quad + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

   iii) $1.2.3 + 2.3.4 + 3.4.5 + ... \quad + n(n+1)(n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$

   iv) $1^2 + 2^2 + 3^2 + ... \quad + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

   v) $1^3 + 2^3 + 3^3 + ... \quad + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$

   vi) Hint: you may want to look back at how you did part (v)

   $$1^4 + 2^4 + 3^4 + ... \quad + n^4 = \frac{1}{30} n(n+1)\left(6n^3 + 9n^2 + n - 1\right)$$

   vii) $1.3.5 + 2.4.7 + 3.5.9 + ... \quad = \dfrac{n(n+1)}{6}\left\{3n^2 + 17n + 25\right\}$

   viii) $\dfrac{1}{1.2} + \dfrac{1}{2.3} + ... \quad + \dfrac{1}{n(n+1)} = \dfrac{n}{(n+1)}$

Exercise 4.2 Solutions

1. i) by direct addition $2+3+4+5+6=20$.

   With $S_n = \frac{n}{2}\{2a+(n-1)d\}$,

   $a=2,\ \ d=1\ \ n=5 \Rightarrow S_5 = \frac{5}{2}(2.2+4.1)=20$

   ii) by direct addition $15+12+9+6=42$.

   Now $a=15,\ \ d=-3\ \ n=4 \Rightarrow S_4 = \frac{4}{2}(2.15+3.-3)=42$.

2. i) $a=3,\ \ d=4$, 8^{th} term $=31$, n^{th} term $=4n$-1, $S_{12}=300$

   ii) $a=4,\ \ d=-2$, 8^{th} term = -10, n^{th} term = 6-2n, $S_{12}=-84$

   iii) $a=1/2,\ \ d=1$, 8^{th} term = 15/2, n^{th} term = n-1/2, $S_{12}=72$

   iv) $a=20,\ \ d=-1$, 8^{th} term = 13, n^{th} term = 21-n, $S_{12}=174$

3. i) $a=1,\ \ d=\frac{3}{8}=0.375$, 9^{th} term $=4$, $S_9=22.5$

   ii) 2050

   iii) 15, 23, 31, 39, 47

4. i) $a=2,\ \ d=2,\ \ S_{24}=600$

   ii) $a=12,\ \ d=4,\ \ S_{16}=672$

   iii) $a=26,\ \ d=-4,\ \ S_{10}=80$

   iv) $a=22,\ \ d=3,\ \ S_{13}=520$

   v) $a=38,\ \ d=-7,\ \ S_9=90$

5. Own series. It is always good to invent problems and then solve them.

6. Show but look back to Exercise 2.1 No.2

7. Use $S_n = \dfrac{a(1-r^n)}{1-r}$

i) $a=2, \quad r=2, \quad n=5, \quad S_5 = 62$

ii) $a=9, \quad r=1/2, \quad n=5, \quad S_5 = \dfrac{279}{16}$

iii) $a=5, \quad r=1/3, \quad n=5, \quad S_5 = \dfrac{605}{81}$

8. i) $a=1, \quad r=1/2, \quad n=4, \quad S_4 = \dfrac{15}{8}$

ii) $a=1, \quad r=1/2, \quad n$ terms, $S_n = 2\left(1-\dfrac{1}{2^n}\right)$

iii) $a=1/2, \quad r=2, \quad n=5, \quad S_8 = \dfrac{255}{2}$

9. Own geometric series.

10. Note that there are $(n+1)$ therefore $S_{n+1} = \dfrac{a(1-r^{n+1})}{1-r} = \dfrac{1-x^{n+1}}{1-x}$

11. i) G.P. $a=5, \quad r=1/3, \quad |r|<1 \Rightarrow S_\infty = 15/2 = 7.5$

ii) G.P. $a=1, \quad r=1/5, \quad |r|<1 \Rightarrow S_\infty = 1.25$

12. i) Geometric series which converges for a finite number of terms with $a=1, \quad r=x$ and $S_n = \dfrac{1(1-x^n)}{1-x}$. If $|x|<1$ then the infinite series converges since $x^n \to 0$ as $n \to \infty$ and thus $S_\infty = \dfrac{1}{1-x}$. If $|x|>1$ then the series diverges. $|x|=1$ needs further investigation. When $x=1$ the series clearly diverges. For $x=-1$ the partial sums oscillate between 0 and 1 hence does not converge, i.e diverges.

ii) Geometric series which converges for a finite number of terms with $a=1,\ \ r=-x$ and $S_n=\dfrac{1(1-(-x)^n)}{1-(-x)}$. If $|x|<1$ then the infinite series converges since $(-x)^n \to 0$ as $n\to\infty$, while $S_\infty=\dfrac{1}{1+x}$. If $|x|>1$ then the series diverges. $|x|=1$ needs further investigation as above.

13. i) $\displaystyle\sum_{k=1}^{5}\frac{1}{2k-1}$ ii) $\displaystyle\sum_{k=1}^{5}\frac{(-1)^{2k+1}}{2k-1}$

iii) $\displaystyle\sum_{k=1}^{6}\frac{(-1)^k}{2(k+1)}$

14. Recall $\dfrac{1}{1+x}=1-x+x^2-x^3+\ldots.$ which converges when $|x|<1$

i) $\dfrac{1}{1+2x}=1-2x+2^2x^2-2^3x^3+\ldots.$ valid when $|x|<1/2$

ii) $\dfrac{1}{1+x^2}=1-x^2+x^4-x^6+\ldots.$ valid when $|x|<1$

iii) $\dfrac{1}{1-x}=1+x+x^2+x^3+\ldots.$ valid when $|x|<1$

iv) $\dfrac{1}{1-3x}=1+3x+3^2x^2+3^3x^3+\ldots.$ valid when $|x|<1/3$

v) $\dfrac{1}{1-\sqrt{x}}=1+x^{1/2}+x+x^{3/2}+\ldots.$ valid when $|x|<1$

15. The Harmonic mean is defined as H_n where $\dfrac{1}{H_n}=\dfrac{\left(\dfrac{1}{a_1}+\dfrac{1}{a_2}+\ldots\ldots+\dfrac{1}{a_n}\right)}{n}$

16. The 10^{th} term is approximately 0.002 while the sum to 10 terms is very nearly 27.

17. $S_\infty=\dfrac{1}{1+3x}$ provided that $|x|<1/3$, otherwise the series does not converge.

18. The series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ diverges.

19. You can sum the series this way but it converges very slowly. There appears to be no easy formula, and indeed if you knew that the answer is $\frac{\pi^2}{6}$ then you might understand why there is no easy formula.

20. The original series is not a GP but if you consider the sum to n terms, S_n, and xS_n then $S_n = \frac{1-x^n}{(1-x)^2} - \frac{nx^n}{(1-x)}$ so as $n \to \infty$ then $S_n \to \frac{1}{(1-x)^2}$ which is a binomial expansion.

Exercise 4.3 Solutions

1. i) $0.090909.... = \frac{9}{10^2} + \frac{9}{10^4} + \frac{9}{10^6} + ...$

This is a geometric series with $a = \frac{9}{10^2}$, $r = \frac{1}{10^2}$, $|r < 1| \Rightarrow S_\infty = \frac{1}{11}$

ii) $0.009009009.... = \frac{9}{10^3} + \frac{9}{10^6} + \frac{9}{10^9} + ...$

This is a geometric series with $a = \frac{9}{10^3}$, $r = \frac{1}{10^3}$, $|r < 1| \Rightarrow S_\infty = \frac{1}{111}$

iii) $0.121212.... = \frac{12}{10^2} + \frac{12}{10^4} + \frac{12}{10^6} + ...$

This is a geometric series with $a = \frac{12}{10^2}$, $r = \frac{1}{10^2}$, $|r < 1| \Rightarrow S_\infty = \frac{4}{33}$

iv) $2.0676767.... = 2 + \frac{67}{10^3} + \frac{67}{10^5} + \frac{67}{10^7} + ...$

This is 2 + (a geometric series) with

$$a = \frac{67}{10^3}, \quad r = \frac{1}{10^2}, \quad |r < 1| \Rightarrow S_\infty = \frac{67}{990}$$

Hence $2.0676767.... = \frac{2047}{990}$

Exercise 4.4 Solutions

1. Usual proof by induction. Note that you will have already proved these by a construction method.

Exercise 4.5 Solutions

Given a statement or result involving an integer n. We need a start and so we first show that result to be true for some $n = k$, usually $k = 1$ but this can vary in some problems. Then we assume the result to be true for but for some unspecified integer $n = m$ say, and we try to show that under this assumption the result is true for $n = m + 1$. Hence, by induction, true for all $n \ge k$.

1. Usual induction. However in part (vii) note that you can write

$$\frac{1}{1.2.3} = \frac{1}{2}\left\{\frac{1}{1.2} - \frac{1}{2.3}\right\}$$

Try and generalise this result to find the sum of the series as

$$\frac{1}{4}\frac{n^2 + 3n}{(n+1)(n+2)}$$

2. To show that $5^{2n} - 6n + 8$ is a multiple of 3 for all positive integers n.

$n = 1 \Rightarrow 5^{2n} - 6n + 8 = 27 = 9 \times 3$, so the result is true for $n = 1$.

Assume the result is true for some $n = m$, say. Then $5^{2m} - 6m + 8 = k \times 3$, for some integer k. Now consider the result for $n = m + 1$ and show that

$$5^{2((m+1)} - 6(m+1) + 8 = (25k + 48m - 66) \times 3$$

which is a multiple of 3 and the result is true for $n = m + 1$.

Hence true for all positive integers.

3 . To find the sum $\Sigma = 1^2 + 3^2 + 5^2 + \ldots\ldots + (2n-1)^2$ add and take away the even terms, i.e.

$$\Sigma = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots\ldots + (2n-1)^2$$
$$-2^2 - 4^2 - \ldots$$

Use the result proved by induction in Question 1 (i) above that

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots\ldots + (n)^2 = \frac{n}{6}(n+1)(2n+1)$$

replacing n by $2n$-1 to give

$$\Sigma = \frac{n}{3}(2n-1)(4n-1) - 4\left\{\frac{n}{6}(n-1)(2n-1)\right\} = \frac{n}{3}(2n-1)(2n+1)$$

For n = 1, 2, 3 calculate the sum by direct addition and then use the formula just proved to show

$n = 1 \quad \Sigma = 1$

$n = 2 \quad \Sigma = 10$

$n = 3 \quad \Sigma = 35$

4. Note that here we use the 'dot' notation for multiplication. That is, $2.3 = 2 \times 3 = 6$

6. $1.1 + 3.2 + 5.2^2 + 7.2^3 + \ldots \quad \ldots + (2n+1).2^n = A + (B + nC)2^n$

We need 3 equations to solve for the 3 unknowns. Try $n = 1$, 2 and 3 and solve the equations to give $A = 3$, $B = -2$, and $C = 4$. Check that these satisfy the equations before proceeding, a good routine to get into.

To prove the result use induction. The result is true for $n = 1$ since $1.1 + 3.2 = 7$. Now assume that the result is true for $n = m$, say, i.e. this means that

$$1.1 + 3.2 + 5.2^2 + 7.2^3 + \ldots \quad \ldots + (2m+1).2^m = A + (B + mC)2^m$$

and by adding the next term $(2[m+1]+1)2^{m+1}$ to both sides show the result is true for $n = m+1$, and hence true for all positive integers.

5. Usual induction proof. This can be used to show that $\sqrt{2}$ is irrational.

6. Usual induction proof.

7. There are of course different ways but one uses the result that $n^2 > 2$ for $n \geq 2$ to show that $2k^2 > (k+1)^2$ and hence the result follows.

8. Write out the result for $n = m+1$ and then use the result from Exercise 2.1 No.2 part ii) i.e. $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ and thus write the series for $n = m+1$ as two times the series for m for which we assume that the result holds.

9. Usual induction but think what also happens if all the numbers are negative.

10. For i) use the binimial theorem for the result with $n = m+1$.

11. You will need to use the binomial expansion of A^{m+1}.

12. From $A_n \geq G_n$ we know that $\frac{a+b}{2} \geq (ab)^{1/2}$ but the result also tells us that equality only occurs when $a = b$.

13. Write $\frac{a+bx^4}{x^2}$ as the sum of two positive numbers and then use the result that $A_n \geq G_n$ with equality only holding when the numbers are equal, i.e. $x^4 = \frac{a}{b}$.

14. Use induction to show that all squares of odd numbers when divided by 4 and 8 leave a remainder 1. Note that this does not imply that all odd numbers whose remainder is 1 when divided by 4 and 8 are perfect squares, e.g. try 41. To do the remainder then use the formula to develop an expression and use the above result to give a contradiction.

15. Usual induction proof. You might wish to use the trick of adding something and taking it away in your working.

$$\lim_{a \to b} \left\{ \frac{a^n - b^n}{a-b} \right\} = nb^{n-1}$$

which is how we establish the derivative of a function $y = x^n$.

We have show that $x^n - 1$ has a factor $(x-1)$. We may rewrite

$$x + x^3 + x^9 + x^{27} + x^{81} = x - 1 + x^3 - 1 + x^9 - 1 + x^{27} - 1 + x^{81} - 1 + 5$$

and so we see the amswer is 5.

16. The result is not true for $n = 1,2,3,4,5$ but is true for $n = 6$. Also recall from the notes that we will have shown that $2 < \left(1+\frac{1}{n}\right)^n < 3$. If you cannot recall this then try and prove it.

17. Not true for $n = 1$, since we have equality but you can show by induction that it is true for all $n \geq 2$. Hence $\frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times .... \times \frac{99}{100} < \frac{1}{\sqrt{151}}$

Exercise 4.6 Solutions

1. These all tend to infinity as $n \to \infty$ and so they all diverge.

2. i) For powers of 2 the partial sum can be shown to satify $n = 2^m$ then

$S_n > 1 + m\frac{1}{2}$ and since $\left\{1 + m\frac{1}{2}\right\} \to \infty$ as $n \to \infty$, then S_n also diverges.

ii) $S_n > \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \ldots. + \frac{1}{\sqrt{n}} = n\frac{1}{\sqrt{n}} = \sqrt{n} \to \infty$ and so diverges.

Exercise 5.1 Solutions

1. $f(x) = x^2$

   i) $f(B) = B^2$

   ii) $f(\alpha) = \alpha^2$

   iii) $f(x^2) = \left(x^2\right)^2 = x^4$

   iv) $f(x+1) = (x+1)^2 = x^2 + 2x + 1$

   v) $f(e^x) = \left(e^x\right)^2 = e^{2x}$

   vi) $f(x+y) = (x+y)^2 = x^2 + 2xy + y^2$

   vii) $f(9) = 9^2 = 81$

2. $f(x) = x^3 + 1$

   i) $f(B) = B^3 + 1$

   ii) $f(\alpha) = \alpha^3 + 1$

   iii) $f(x^2) = \left(x^2\right)^3 + 1 = x^6 + 1$

   iv) $f(x+1) = (x+1)^3 + 1 = x^3 + 3x^2 + 3x + 1 + 1 = x^3 + 3x^2 + 3x + 2$

   v) $f(e^x) = \left(e^x\right)^3 + 1 = e^{3x} + 1$

   vi) $f(x+y) = (x+y)^3 + 1 = x^3 + 3x^2y + +3xy^2 + y^3 + 1$

   vii) $f(9) = 9^3 + 1 = 729 + 1 = 730$

3. This functions is very easy to describe but rather hard to write down. Once such function is $f(n) = \left(1 + (-1)^n\right)\frac{n}{2}$ but this is not unique.

4. $f(x)$ is linear if for two numbers x_1 and x_2 then if $f(x_1) = f_1$ and $f(x_2) = f_2$ we have $f(x_1 + x_2) = f_1 + f_2$

Exercise 5.2 Solutions

1. i) Yes , $f(x): R \to R$ so domain and codomain are the set of real numbers.

   ii) Yes, provided that $x \neq 1$. If $R^{-} = \{x \in R \textit{ such that } x \neq 1\}$ then

   $f(x): R^{-} \to R^{-}$

   iii) Not single valued so not a function.

   iv) Not single valued so not a function.

   v) Yes, $f(x): R \to R$ but range of $y = f(x) \geq -1$

   vi) Yes, provided that $x \neq 0$. If $R^{-} = \{x \in R \textit{ such that } x \neq 0\}$ then

   $f(x): R^{-} \to R^{-}$

2. i) For $f(x)$ to be real we require $-3 \leq x \leq 3$

   ii) For $f(x)$ to be real we require $-2 \leq x \leq 2$

   iii) $f(x)$ not defined for $x = 7, -2$

   iv) $f(x)$ not defined for $x = 3, -2$

   v) $2 \leq x \leq 5$

   vi) No possible values

3. i) Function not defined for $x = -1$. Range is such that y cannot take value of $y = 0$ when $x = -1$

   ii) Sketch

   iii) $f(2x) = \dfrac{1}{1+2x}, \quad x \neq -1/2$

   $f(2+x) = \dfrac{1}{3+x}, \quad x \neq -3$

   $f(1/x) = \dfrac{x}{x+1}, \quad x \neq -1$

   $f(x^2 - 2) = \dfrac{1}{x^2 - 1}, \quad x \neq -1, 1$

   $f(x+y) = \dfrac{1}{1+x+y}, \quad x + y \neq -1$

   $f(x) + f(y) = \dfrac{1}{1+x} + \dfrac{1}{1+y}, \quad x \neq -1, \quad y \neq -1$

   Note $f(x) + f(y) \neq f(x+y)$

4. If the domain is taken as the entire real line then by plotting the functions, or otherwise, give the ranges of the following functions

i) All real values

ii) $y \geq 1$

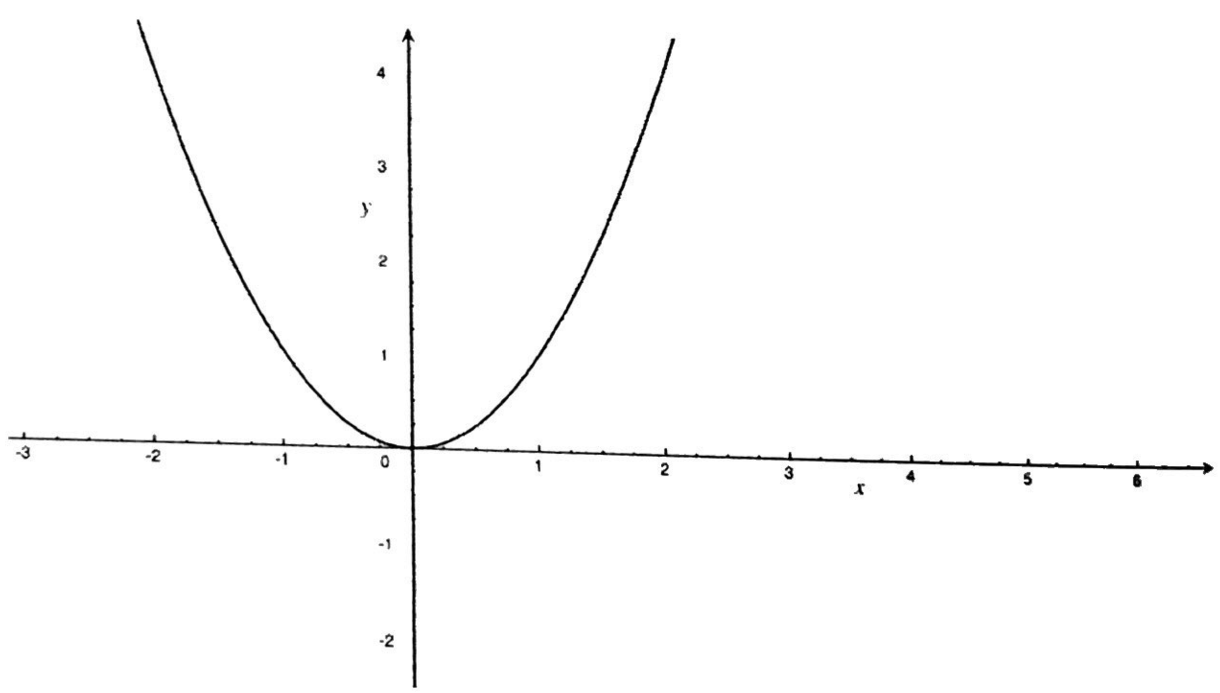
iii) $y \geq 2$

iv) $y \geq -1$

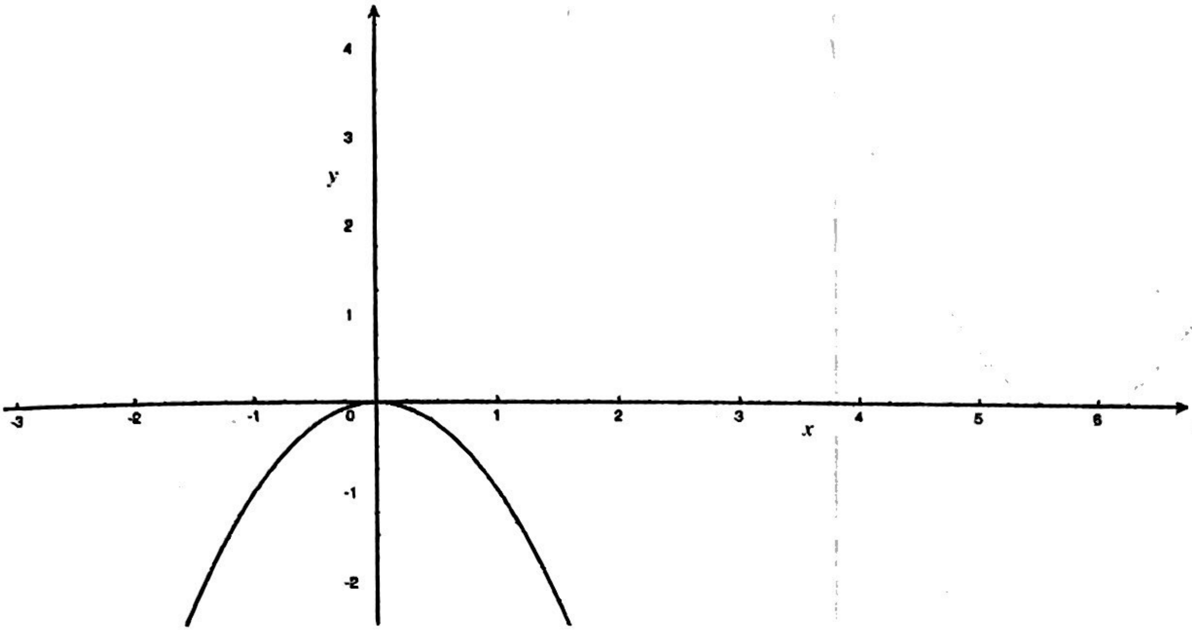
v) $y < 3$

5. These graphs were produced using the Grapher software on my Mac

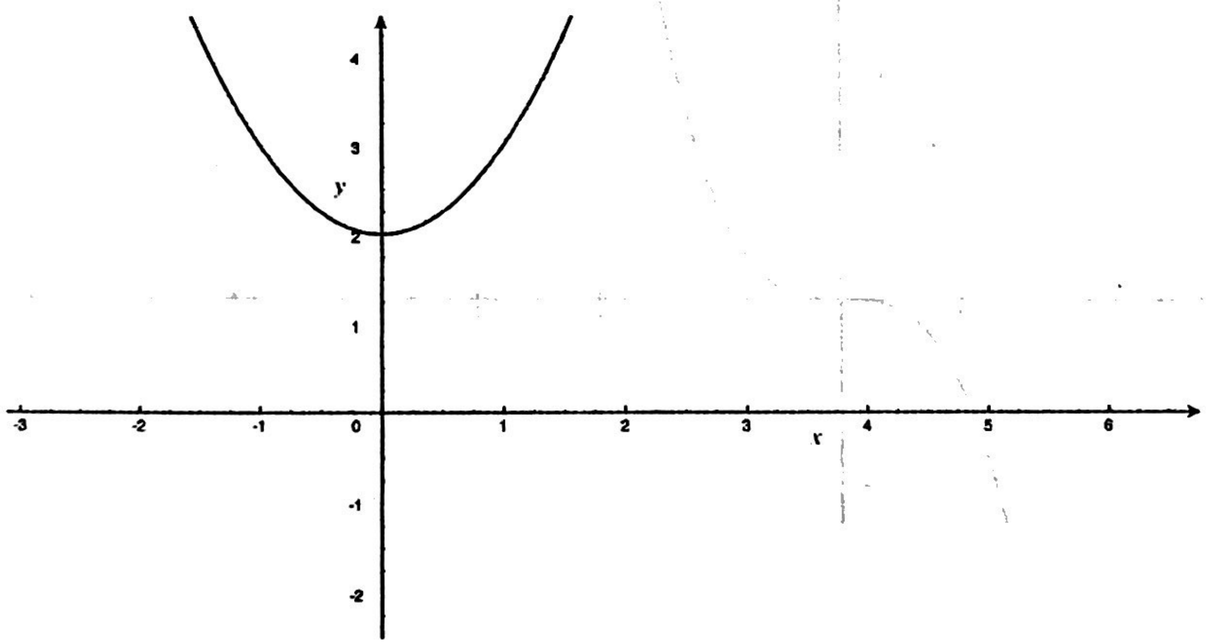
i) $y = f(x) = f(-x) = x^2$



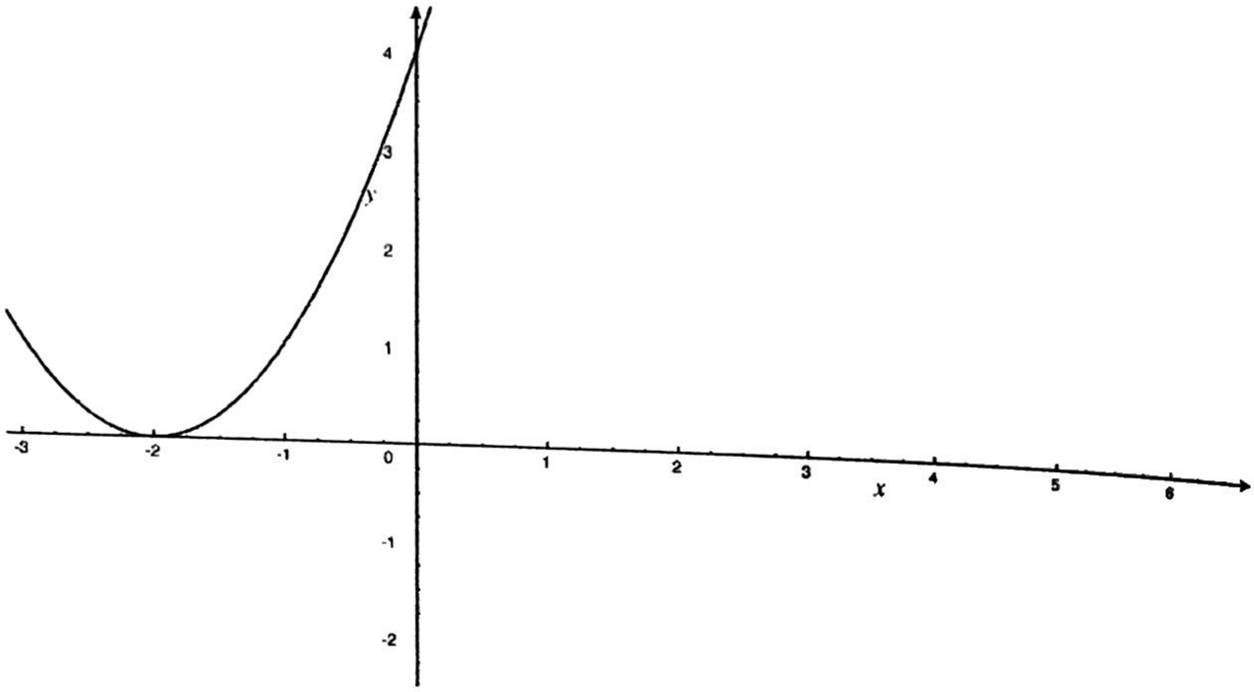
$y = -x^2$



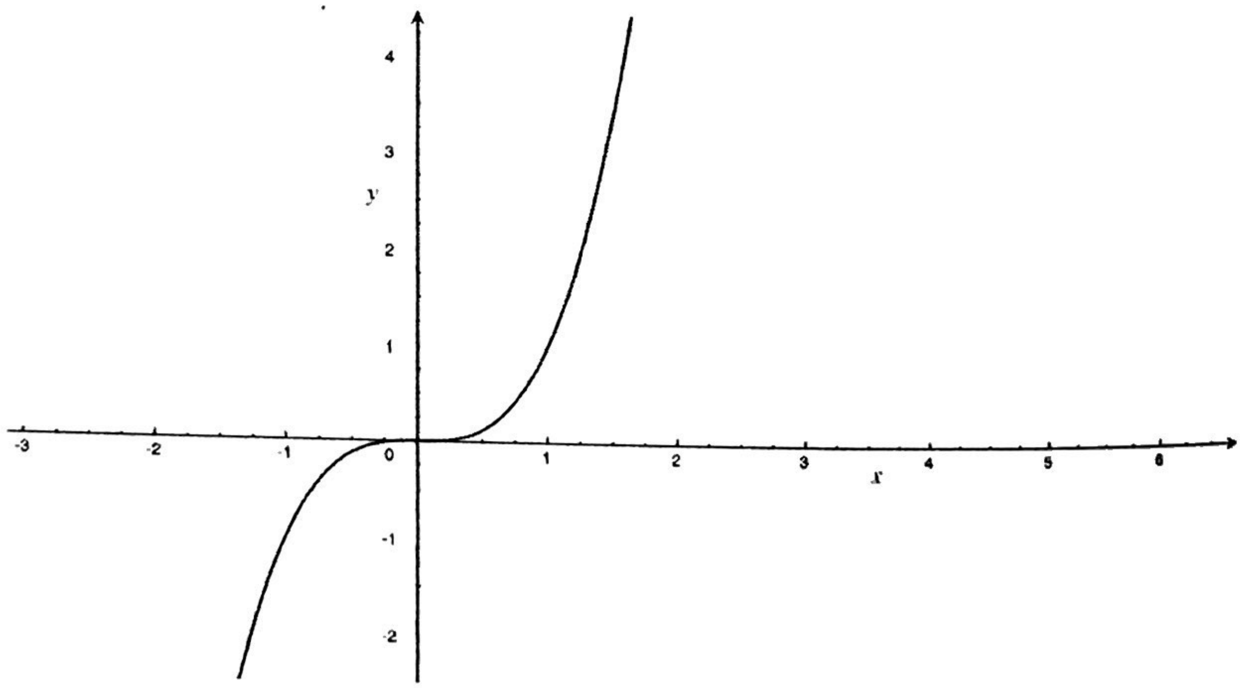
$y = x^2 + 2$



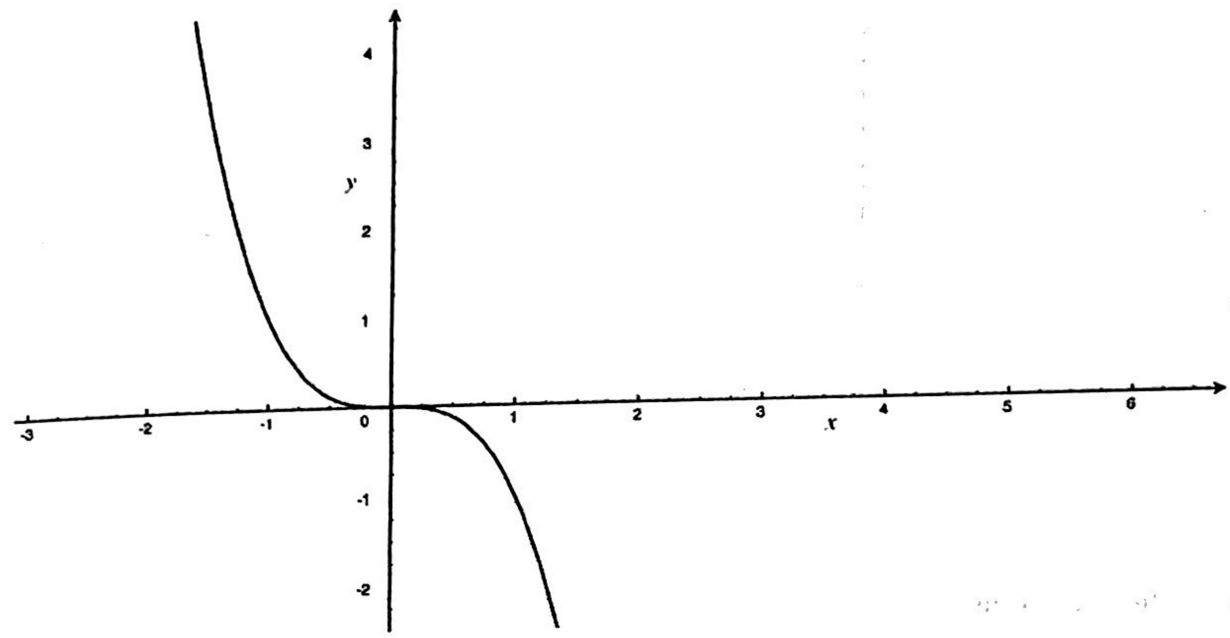
$y = (x + 2)^2$



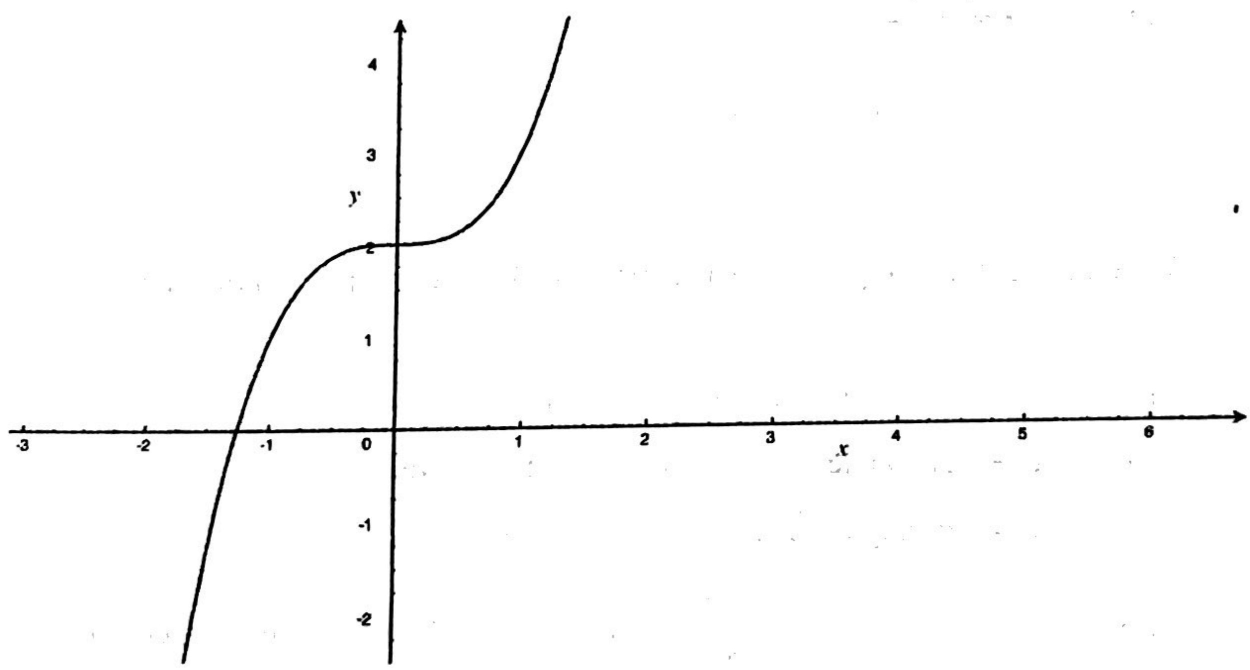
ii) $y = f(x) = x^3$



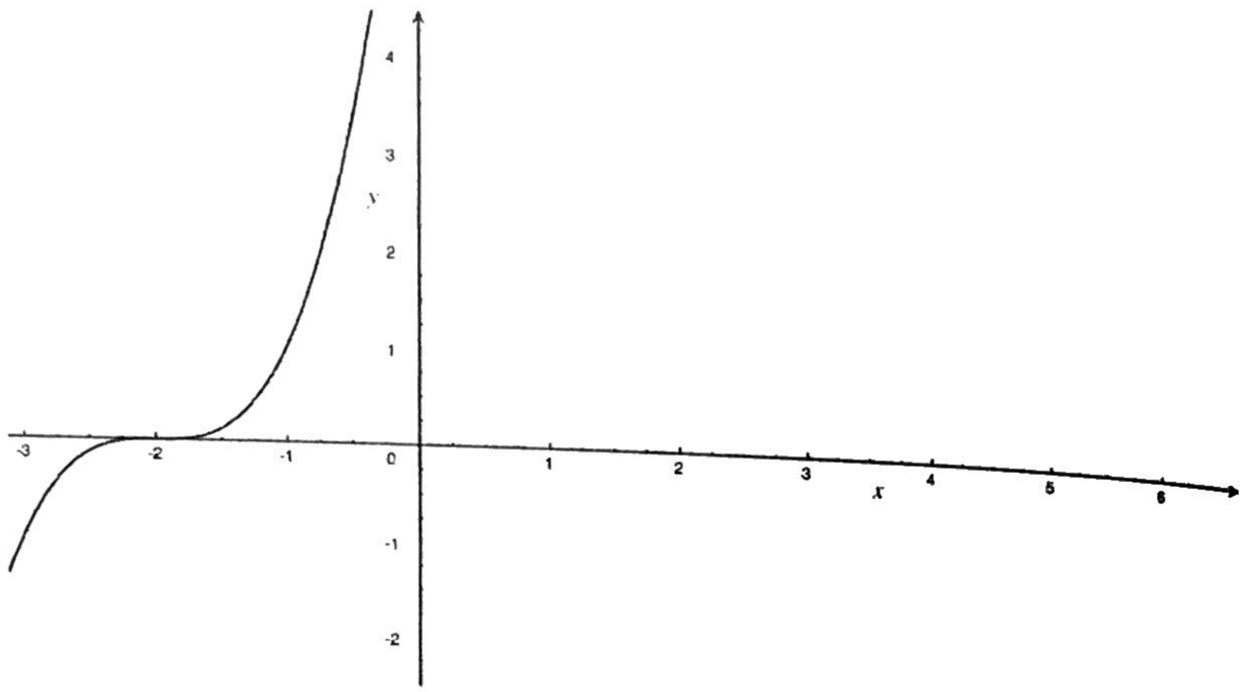
$$y = -x^3 = (-x)^3$$



$$y = x^3 + 2$$



$y = (x+2)^3$



6. Just plug it in.

7. i) $y = \pm\sqrt{8 - x^2} - 1$, need to take the positive square root to define a single valued function

ii) $y = \frac{9}{x}, \quad x \neq 0$

iii) $y = \sqrt[3]{2x - 4}$

iv) $y = \left(\frac{x^5 + 4}{x}\right)^{3/2}, \quad x \neq 0$

8. Which of the following produce a single valued relationship, i.e. a function

i) $y = x^2$ is single valued hence a function

ii) $y^2 = x$ is double valued so strictly not a function

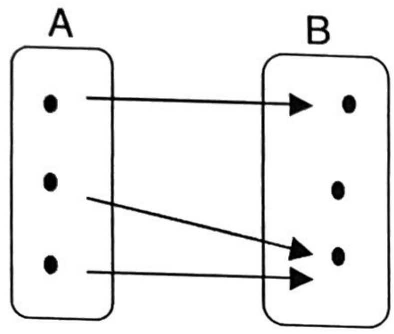
iii) $y^3 = x$ is single valued hence a function

iv) $x^2 + 2y^3 = 3 \quad \Rightarrow y = \sqrt[3]{\frac{3 - x^2}{2}}$ which is single valued since the cube root is unique.

Exercise 5.3 Functions as Mappings

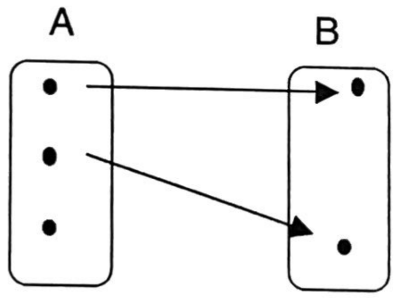
1.

i)



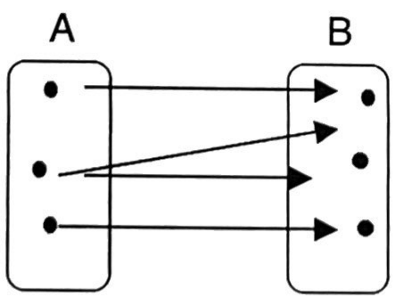
Yes. It does not matter that there is some spare in B as long as all the elements in A are sent somewhere.

ii)



No, there is an element of A for which we do not know where the function sends it.

iii)



No. the function must send an element of A to a *unique* element of B.

Exercise 5.4 Solutions

1. i) $f \circ g = 2x^2$, $g \circ f = 4x^2$ therefore f and g do not commute

ii) $f \circ g = \frac{1}{x^2+1}$, $g \circ f = \frac{1+x^2}{x^2}$ therefore f and g do not commute

iii) $f \circ g = 3(x^2+2)^3$, $g \circ f = 9x^6 + 2$ therefore f and g do not commute

iv) $f \circ g = 8x$, $g \circ f = 8x$ therefore f and g commute

v) $f \circ g = \frac{7x+3}{3x+1}$, $g \circ f = \frac{7x+3}{3x+1}$ therefore f and g commute

vi) $f \circ g = \sin^3(x)$, $g \circ f = \sin(x^3)$ therefore f and g do not commute

vii) $f \circ g = x$, $g \circ f = x$ therefore f and g commute.

2. Let $f(x) = \cos(x)$, $g(y) = y^2$ then

i) $(g \circ f)(x) = \cos^2(x)$

ii) $(f \circ g)(x) = \cos(x^2)$

3. i) $h_1(x) = 3x^3 - 1 = f \circ g(x)$

ii) $h_2(x) = x^3 + 3x - 1 = g(x) + f(x)$

iii) $h_3(x) = 3x^4 - x^3 = g(x) \cdot f(x)$

iv) $h_4(x) = 3x^{12} - x^9 = (g \circ g)(x) \cdot (f \circ g)(x)$

v) $h_5(x) = 9x - 4 = (f \circ f)(x)$

vi) $h_6(x) = 3x^3 - 9x - 4 = (f \circ g)(x) + 3f(x)$

4. i) The domain of f is real line except $x = -1$, the codomain is all real values except $y = 0$

ii) $f(1/x) = \frac{x}{x+1}$, $x \neq -1$ $\quad$ $f(3x) = \frac{1}{1+3x}$, $x \neq -1/3$

$f(x+y) = \frac{1}{1+x+y}$, $x + y \neq -1$

$f(x) + f(y) = \frac{2+x+y}{(1+x)(1+y)}$, $x \neq -1, y \neq -1$

iii) $(f \circ f)(x) = \frac{1+x}{2+x}$, $x \neq -2$

5. i) $2^{3x} = (f \circ h)(x)$

ii) $2^{x^3} = (h \circ f)(x)$

iii) $x^9 = (f \circ f)(x)$

iv) $\sin(2^{3x}) = (g \circ f \circ h)(x)$

v) $\sin(x^3 + 2^x) = g \circ \{f(x) + h(x)\}$

6. i) $2^{2x} = (f \circ g)(x)$

ii) $2^{x^2} = (g \circ f)(x)$

iii) $2^{\sin(x)} = (g \circ h)(x)$

iv) $\sin(2^x) = (h \circ g)(x)$

v) $\sin(x^2) = (h \circ f)(x)$

vi) $\sin^2(x) = (f \circ h)(x)$

vii) $2^{2^x} = (g \circ g)(x)$

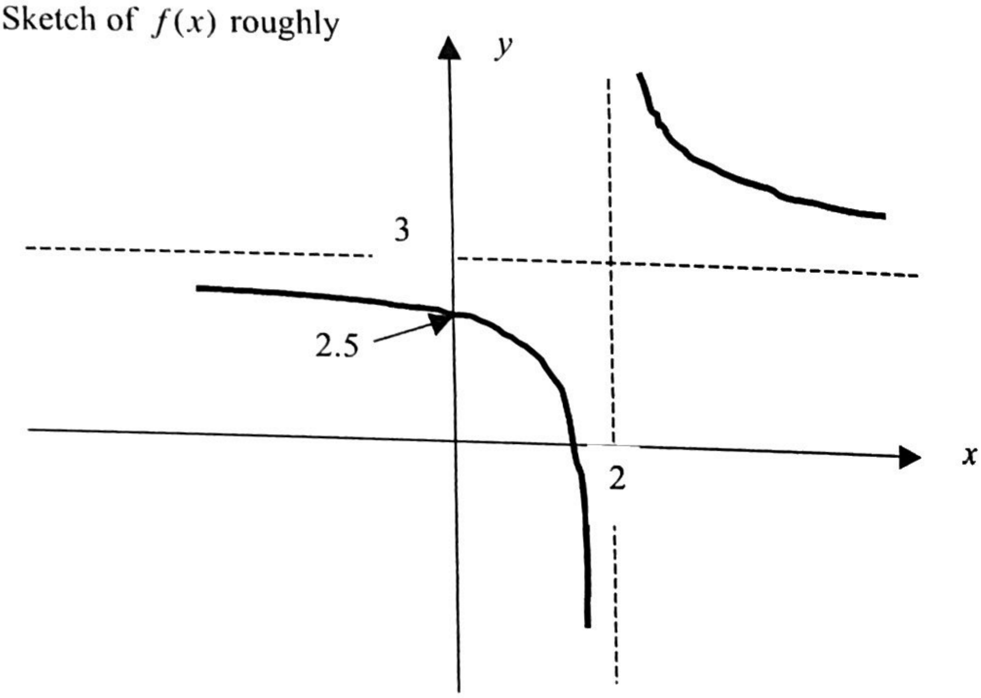
viii) $\sin(y^2 + 2^x) = h \circ \{f(y) + g(x)\}$

ix) $2^{2\sin(x^2)} = (f \circ g \circ h)(x^2)$

Exercise 5.5 Solutions

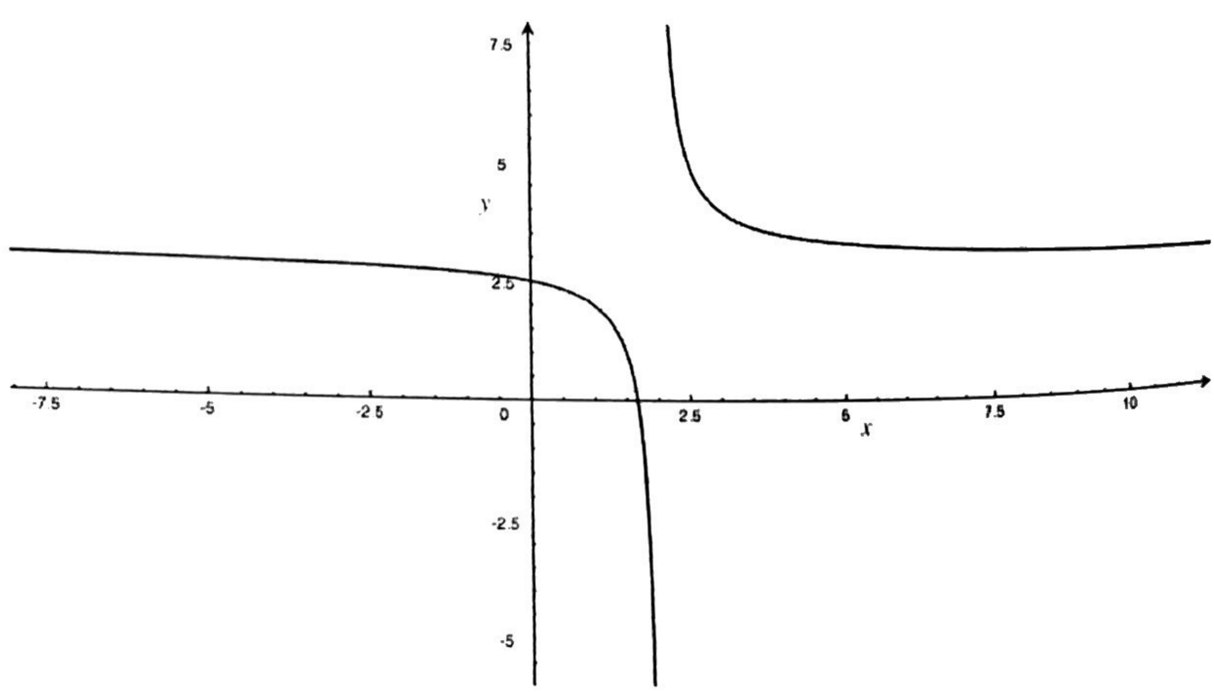
1. Range $\{y \text{ such that } y \in R, \quad y \neq 3\}$. To show that $f(x)$ is 1-1 then assume that x_1, x_2 exist such that $f(x_1) = f(x_2) \quad \Rightarrow x_1 = x_2$. $y = f(x) = \dfrac{1}{x-2} + 3$

Sketch of $f(x)$ roughly



Or the Mac Grapher gives

$$y = f(x) = \frac{1}{x-2} + 3$$



2. i) Function, not onto, not 1-1

   ii) Function, onto, not 1-1

   iii) Not a function

   iv) Not a function

   v) Function, not onto, not 1-1

   vi) Function, not onto, but 1-1

   vii) Function, onto and 1-1

3. If A has m elements and B has n elements then $n > m$ if f is 1-1.
   If f is onto then $n = m$.

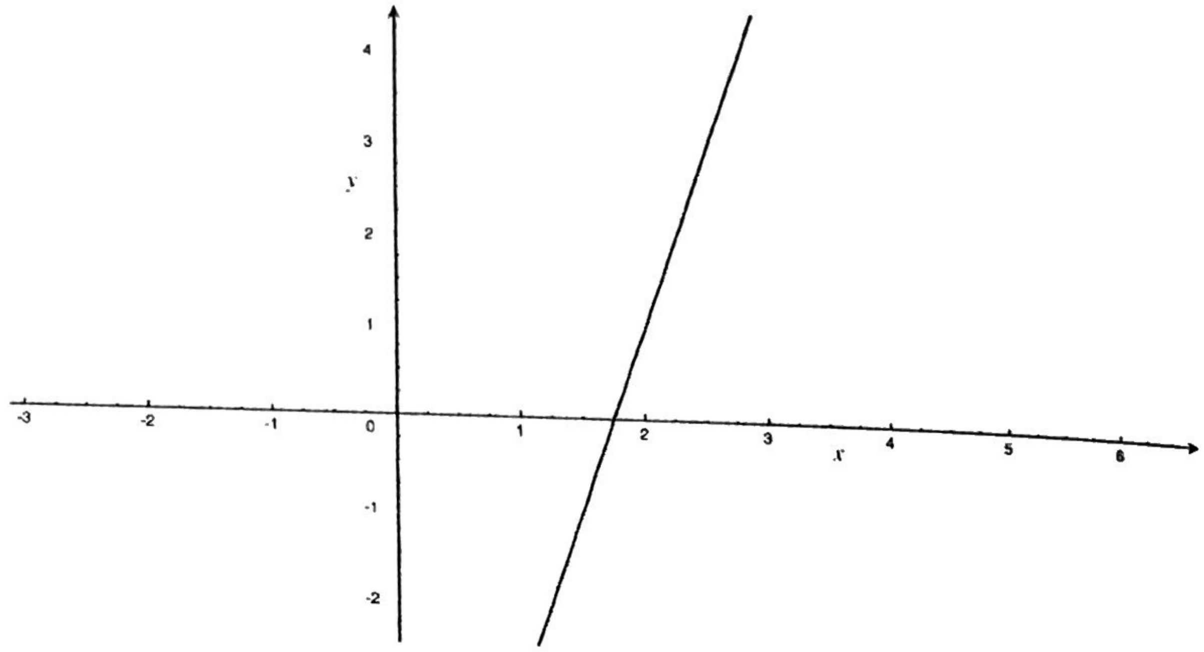
4. i) $g \circ f$ is onto but not 1-1

   ii) $g \circ f$ is not onto and not 1-1

Exercise 5.6 Solutions

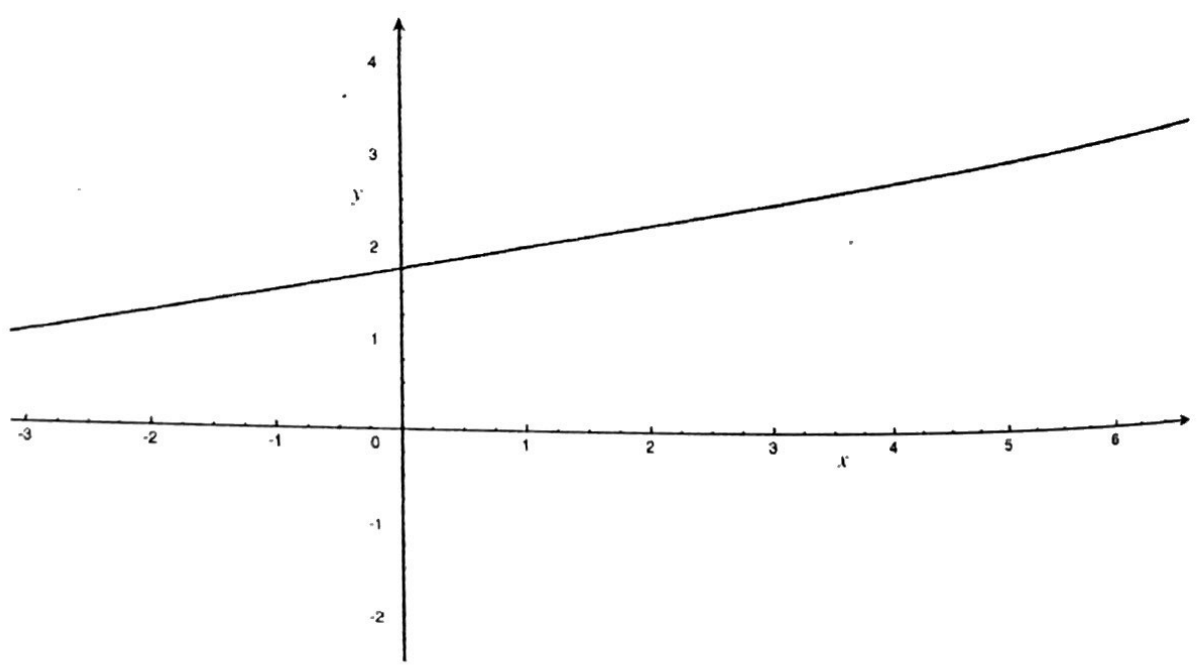
1. Graphs

   i) $f(x) = 4x - 7$

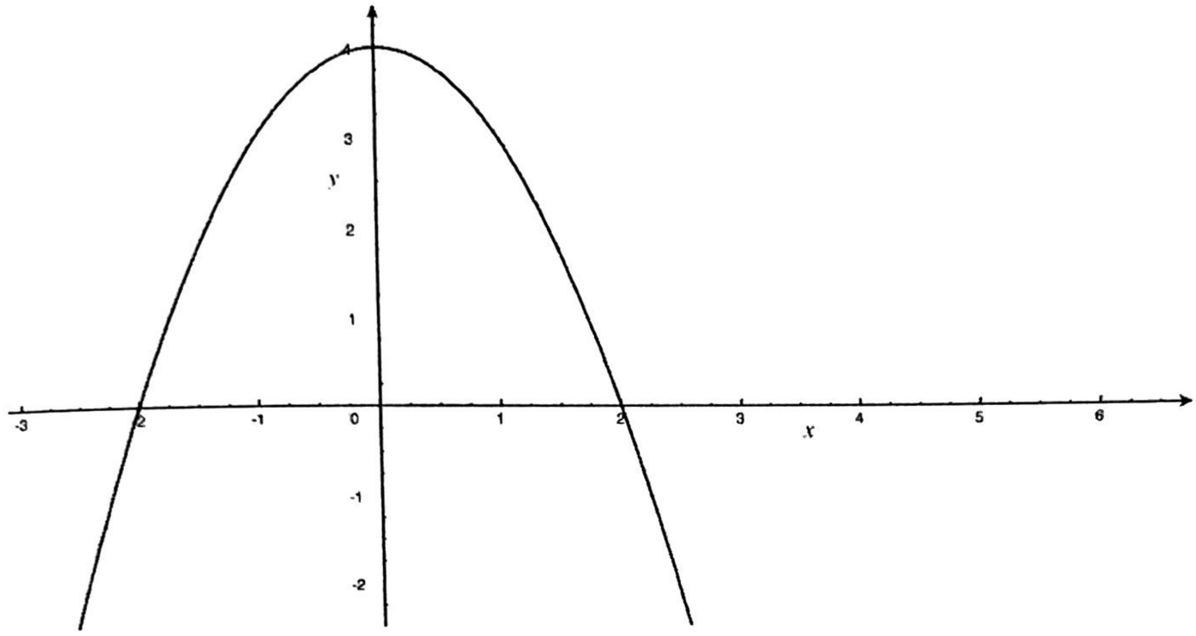


has an inverse $x = \dfrac{y+7}{4}$ or

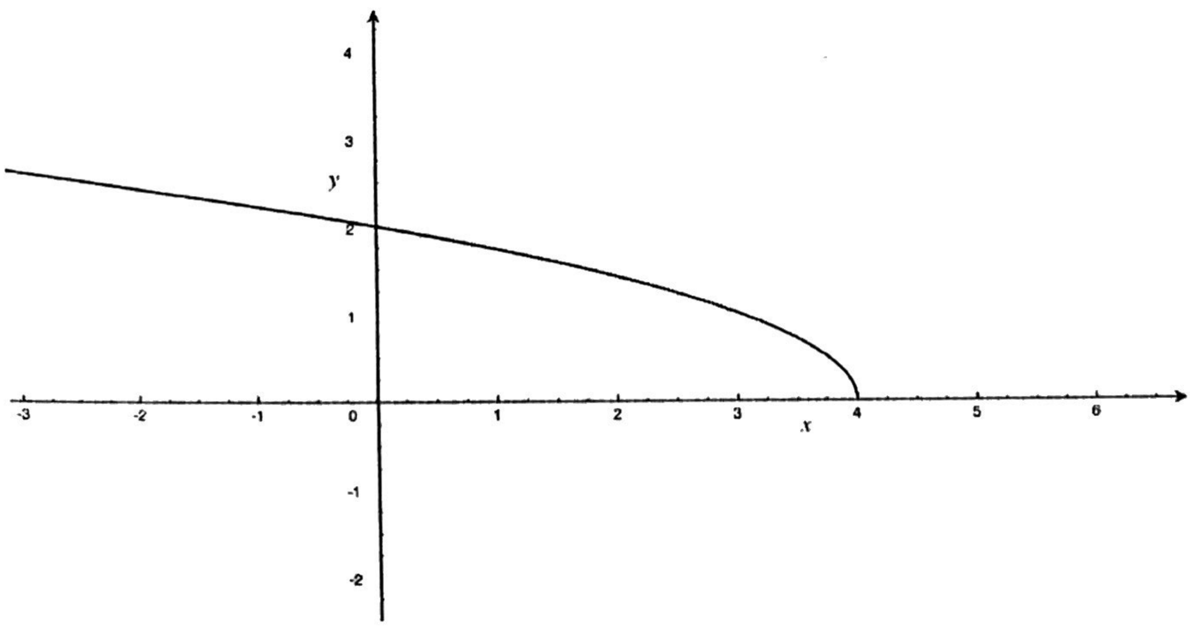
$$y = \frac{x+7}{4}$$



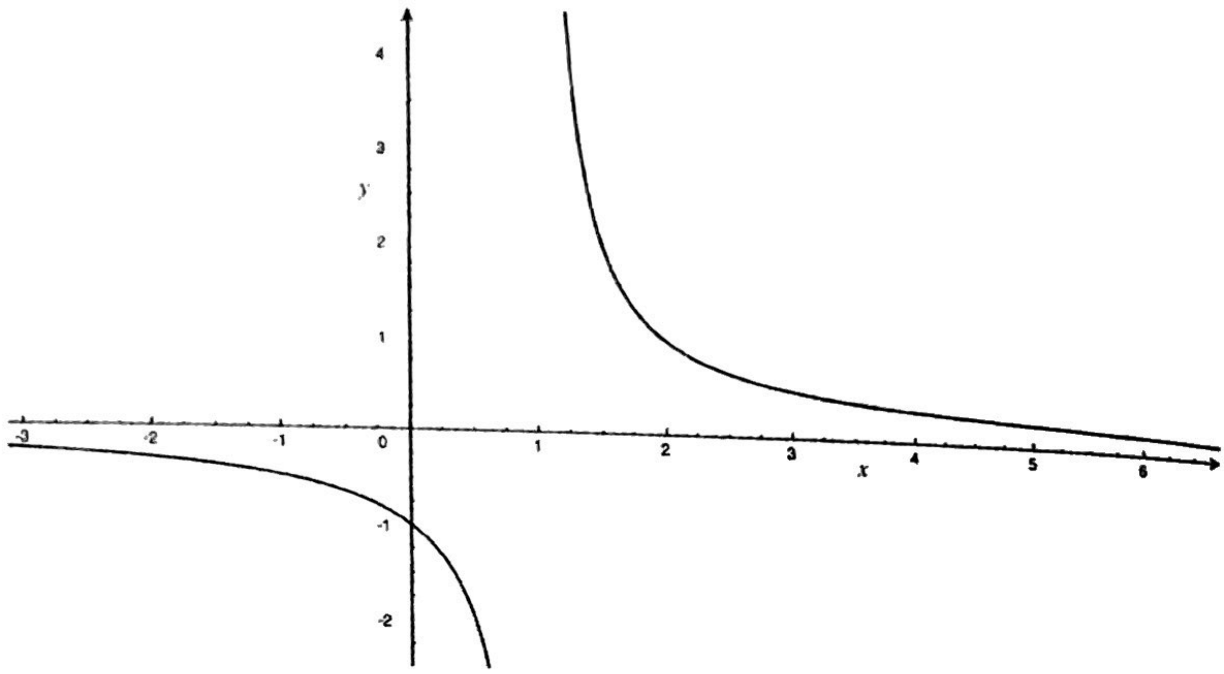
ii) $f(x) = 4 - x^2$ is a function but is not 1-1 and so the inverse relationship is not a function.



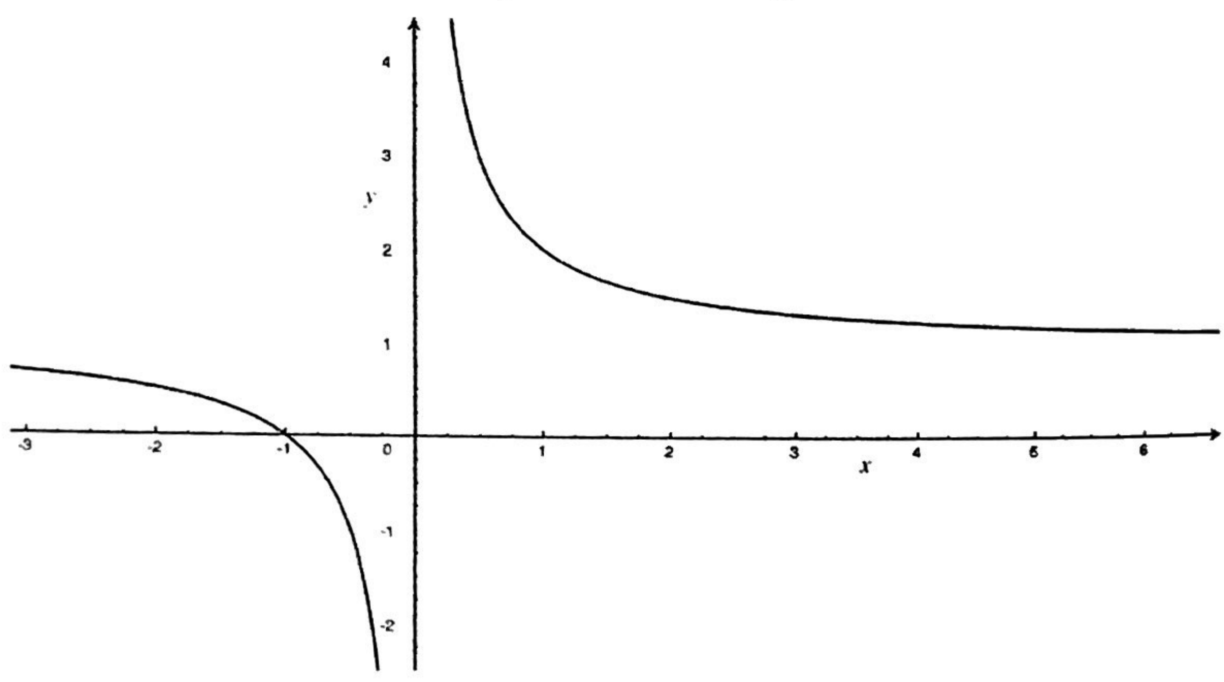
We can of course write $x = \pm\sqrt{4-y}$ and select the positive root as the principal branch and hence define the principal inverse as $F^{-1}(y) = \sqrt{4-y}$ or we can use x and say $F^{-1}(x) = \sqrt{4-x}$



iii) $f(x) = \dfrac{1}{x-1}$ is a function provided that $x \neq 1$.



The inverse is $x = \dfrac{1}{y} + 1$, i.e. $f^{-1}(x) = \dfrac{1}{x} + 1$



2. Only the function defined in (vii) has an inverse.

3. Note that $y = \sqrt{x} = x^{1/2}$ is not in the form $y = x^n$, so we cannot immediately differentiate it. But, think about the definition of an inverse. If $y = \sqrt{x} = x^{1/2}$ then the inverse of y is given by the function $y^{-1} = x^2$. Now since, by definition of an inverse, $y^{-1}\{y(x)\} = x$, we can differentiate this equation to give $\frac{d}{dx} y^{-1}\{y(x)\} = 1$. But using the chain rule, with $u = y(x)$ this means that $\frac{d}{du} y^{-1}\{u\} \frac{d}{dx} y(x) = 1$. This can be rearranged to give the result that we are after, $\frac{d}{dx} y(x) = \frac{1}{\frac{d}{du} y^{-1}\{u\}}$. Note that we ***can*** differentitate y^{-1} since it is in the form $y = x^n$, i.e. $\frac{d}{du} y^{-1}\{u\} = 2u$. Therefore $\frac{d}{dx} y(x) = \frac{1}{2u} = \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-1/2}$.

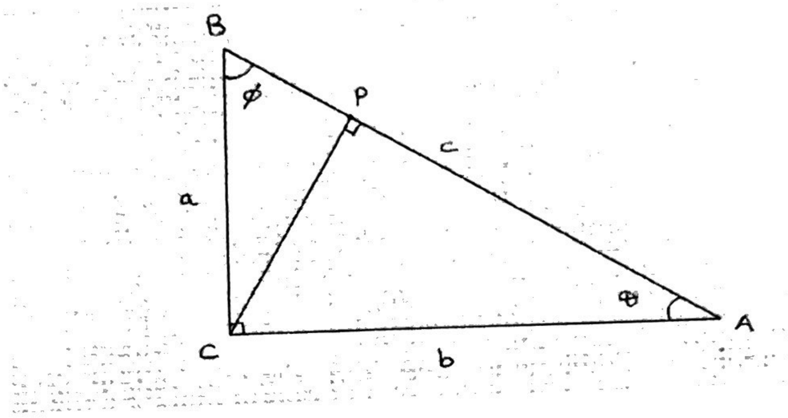
Exercise 6.1 Solutions

1. i) $\pi/6$
   ii) $\pi/3$
   iii) $3\pi/2$
   iv) 3π

2. i) 90
   ii) 15
   iii) 240
   iv) 131.78

Exercise 6.2 Solutions

1. Draw a right-angled triangle. Draw in a line which meets the hypoteneuse at a right angle then consider the cosine of the two acute angles



Then

$$\cos\theta = \frac{b}{c} = \frac{AP}{b} \quad \Rightarrow \quad b^2 = c\,AP$$

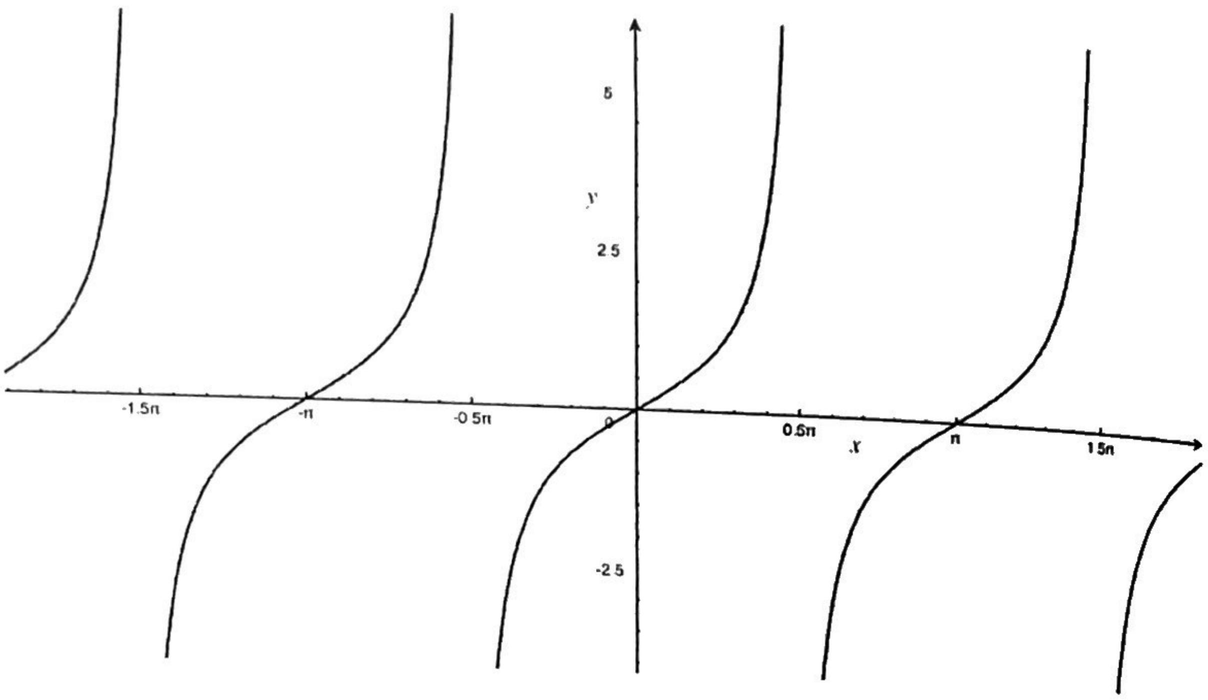
and

$$\cos\phi = \frac{a}{c} = \frac{BP}{a} \quad \Rightarrow \quad a^2 = c\,BP$$

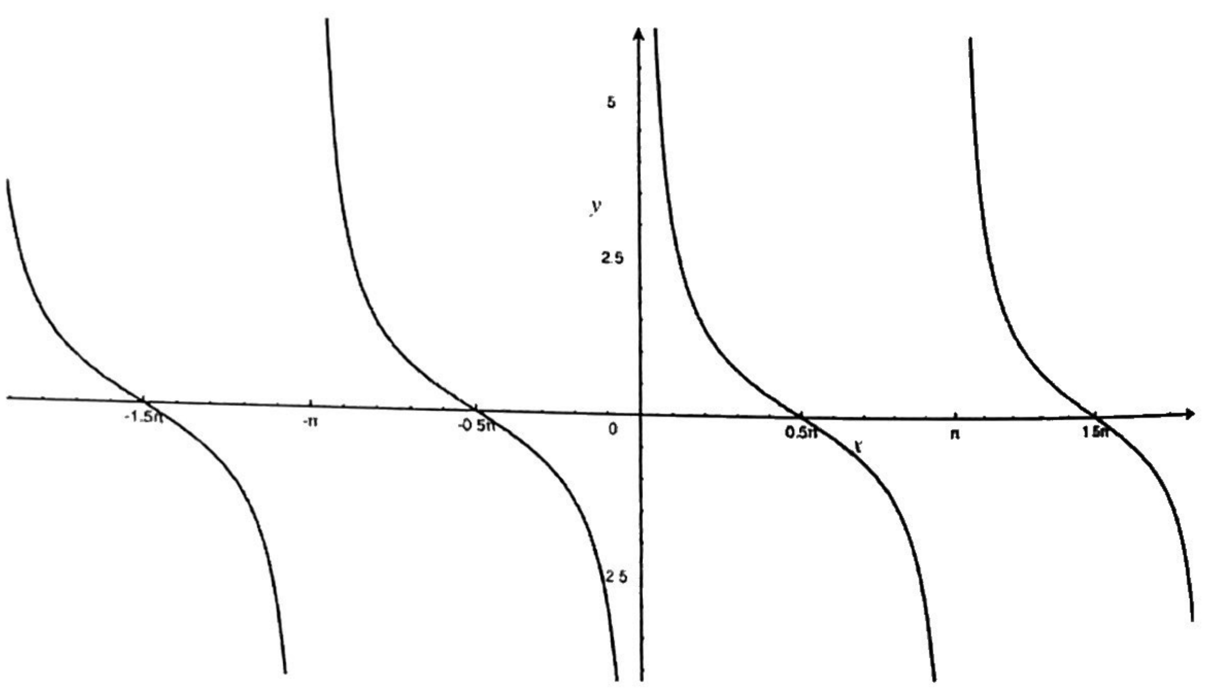
Hence $a^2 + b^2 = c^2$

2. Plot of functions.

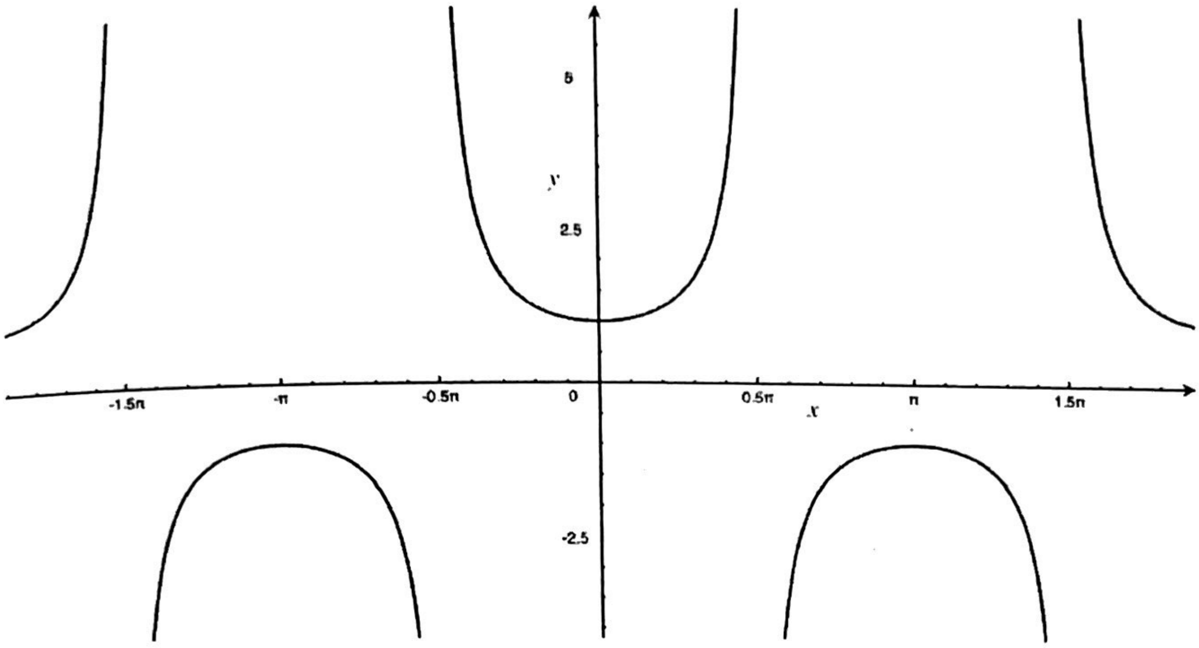
i) $y = \tan(x)$



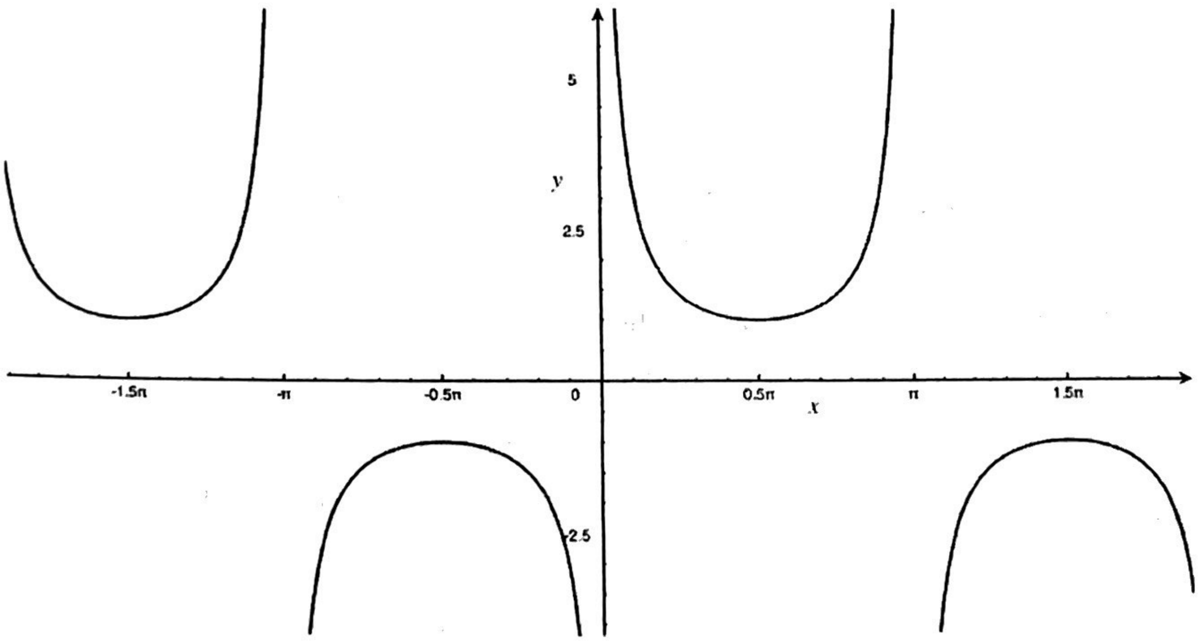
ii) $y = \cot(x)$



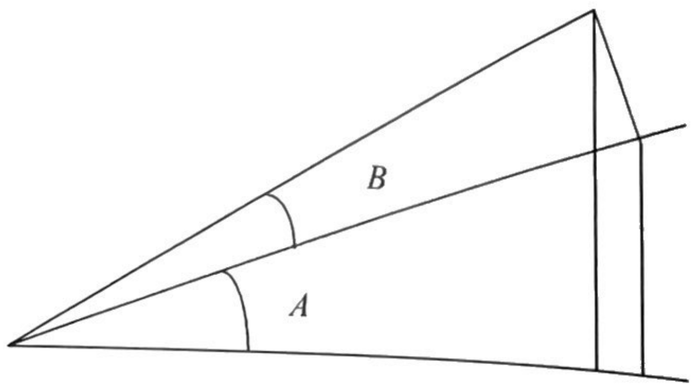
iii) $y = \sec(x)$



iv) $y = \mathrm{cosec}(x)$



3. By geometrical considerations. The key thing is to draw the right picture.



4., $$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B) \qquad (1)$$

Since cos is an even function and sin is odd function then putting $B = -B$ gives

$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B) \qquad (2)$$

i) adding equations (1) and (2) gives the result

ii) subtracting equation (2) from equation (1) yields the result.

In the same way

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B) \qquad (3)$$

$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B) \qquad (4)$$

iii) adding equations (3) and (4) gives the result

ii) subtracting equation (3) from equation (4) yields the result.

5. Considering the graphs of the functions

i) $\sin(170)$ degrees $= \sin(10)$ degrees ≈ 0.17

$\cos(170)$ degrees $= \cos(-10) = -\cos(10)$ degrees ≈ -0.98

$\tan(170)$ degrees $= \tan(-10)$ degrees $= -\tan(10)$ degrees ≈ -0.18

ii) $\sin(-40)$ degrees $= -\sin(40)$ degrees ≈ 0.64

$\cos(-40)$ degrees $= -\cos(40)$ degrees ≈ 0.77

$\tan(-40)$ degrees $= -\tan(40)$ degrees ≈ -0.84

6. Plot graph and consider the product of both tangents.

7. $\cos\theta \times \cos 2\theta \times \cos 4\theta \times .... \times \cos(2^n\theta) = \dfrac{1}{2^{n+1}} \dfrac{\sin(2^{n+1}\theta)}{\sin\theta}$

8. Use results of Question 3 to show that

$$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$$

hence

$$\tan(4x) = \frac{2\tan(2x)}{1-\tan^2(2x)} = \frac{4\tan(x) - 4\tan^3(x)}{1-6\tan^2(x)+\tan^4(x)}$$

9. Angle of the sector is 2 radians ≈ 114.59 degrees. The area of a circle is πr^2

Therefore area of the sector is $\pi r^2 \times \dfrac{2}{2\pi} = r^2$

10. Cosine rule is $c^2 = a^2 + b^2 - 2ab\cos\theta$, where θ is the angle opposite the side c of a triangle. You will have to consider obtuse and acute angles separately.

11. i) The lines joining the three centres forms an equilateral triangle with sides $2R$. The area of this triangle, A, is $A = 2\left\{\dfrac{1}{2} \cdot R \cdot 2R\sin\theta\right\}$

But since it is equilateral $\theta = 60$ degrees $= \dfrac{\pi}{3} \Rightarrow \sin\theta = \dfrac{\sqrt{3}}{2}$ and so $A = \sqrt{3} \cdot R^2$

Meanwhile the area of one sector, S, is $S = \pi R^2 \times \dfrac{\pi/3}{2\pi} = \dfrac{\pi}{6} R^2$

Hence the area enclosed is $A - 3S = \sqrt{3} \cdot R^2 - \dfrac{\pi R^2}{2} = R^2\left\{\sqrt{3} - \dfrac{\pi}{2}\right\} \approx 0.161R^2$

ii) With the fourth disc inserted, consider the angle formed by the line joining the centre of one of the original discs to the centre of the new central disc. This makes an angle of 30 degrees $= \pi/6$ and we know that

$$\cos 30 \text{ degrees} = \cos\left(\pi/6\right) = \frac{\sqrt{3}}{2}$$

From geometrical considerations

$$\cos 30 \text{ degrees} = \cos\left(\pi/6\right) = \frac{\sqrt{3}}{2} = \frac{r}{r+R}$$

$$\Rightarrow r = R\frac{\sqrt{3}}{3}\left(2-\sqrt{3}\right) \approx 0.155R$$

When $R = 2 \Rightarrow r \approx 0.31$ and so a disc of radius 0.5cm would *not* fit in the gap.

12. i) The volume of the cone must be the same as the volume of the original sphere. So if the cone has radius r, height h with a length l from the perimeter of the base to the tip if the cone, then

$$\frac{1}{3}\pi r^2 h = \frac{4}{3}\pi r^3 \quad \Rightarrow h = 4r$$

ii) To find the surface area of the cone First remove the circular base then consider cutting it vertically and opening this out to form a sector whose radius is l and the arc has length $2\pi r$, the circumference of the circular base.

By geometrical consdiderations, using Pythagarus's theorem we can show that

$$l^2 = h^2 + r^2 = 17r^2 \quad \Rightarrow l = r\sqrt{17}$$

If a sector is formed by arc of length x then the angle swept out is $\theta = \dfrac{x}{r}$.

The area of the sector is given by

$$S = \pi r^2 \times \frac{\theta}{2\pi} = \frac{r^2\theta}{2}$$

but in the case that the arc is length x then

$$S = \pi r^2 \times \frac{\theta}{2\pi} = \frac{r^2\theta}{2} = \frac{r^2 x}{2r} = \frac{rx}{2}$$

The area of the sector for the cone with radius $= l$, $x = 2\pi r$ is then $S = l\pi r$, so that

S.A. cone = S.A. of sector + area of the circular base

$$= l\pi r + \pi r^2 = \pi r^2\sqrt{17} + \pi r^2 = \pi r^2\left(1 + \sqrt{17}\right) \approx 5.12\pi r^2$$

This is greater than the surface area of the original sphere which has an area $4\pi r^2$.

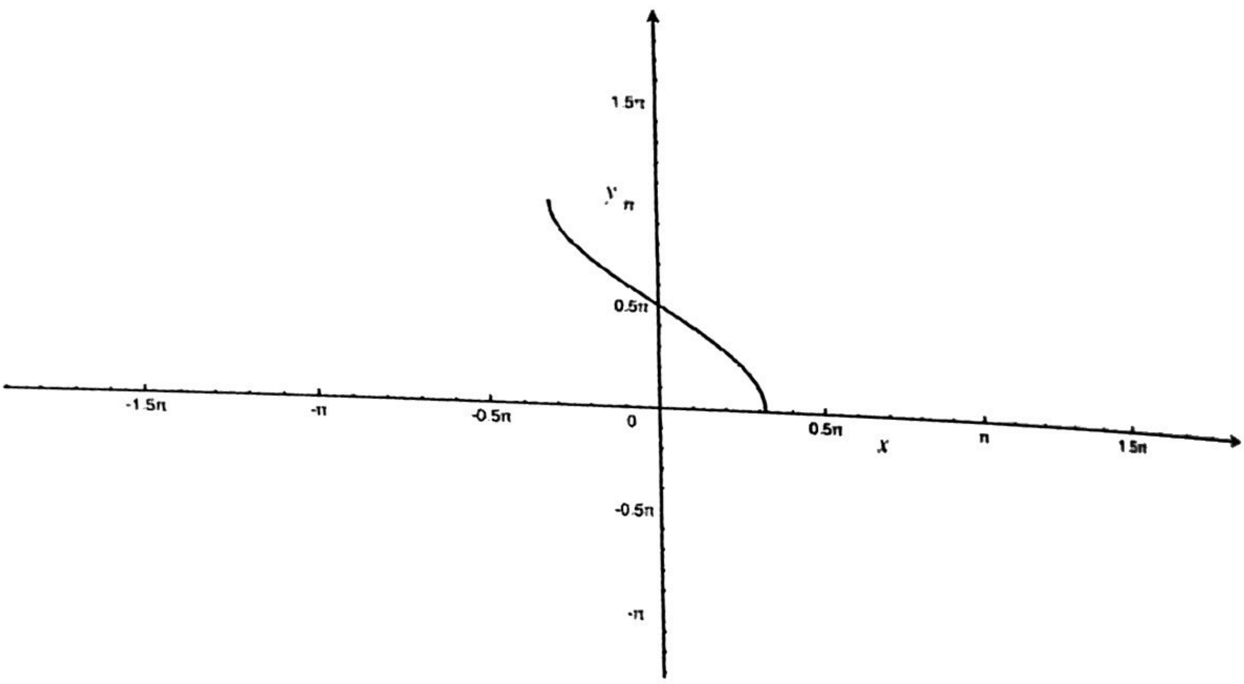
13. For $0 < \theta < \dfrac{\pi}{2}$ then both $\tan\theta$ and $\cot\theta$ are positive. We can use the result that the arithmetic mean is always greater or equal to the geometric mean. The geometric mean in this case is $\sqrt{(\tan\theta \times \cot\theta)} = 1$

14. $\cos(\sin x)$ is bigger but can you show it?

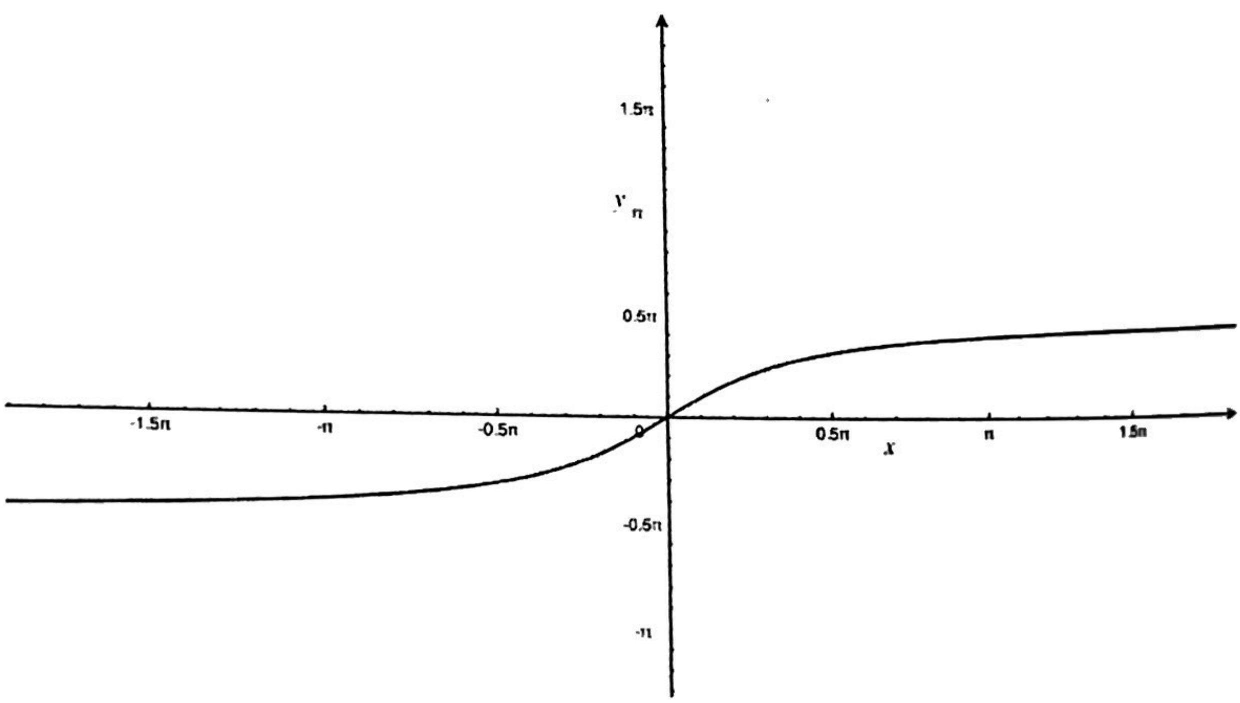
Exercise 6.3 Solutions

1. Plot of functions

i) $y = \text{Cos}^{-1}(x) \Rightarrow 0 < y \le \pi$

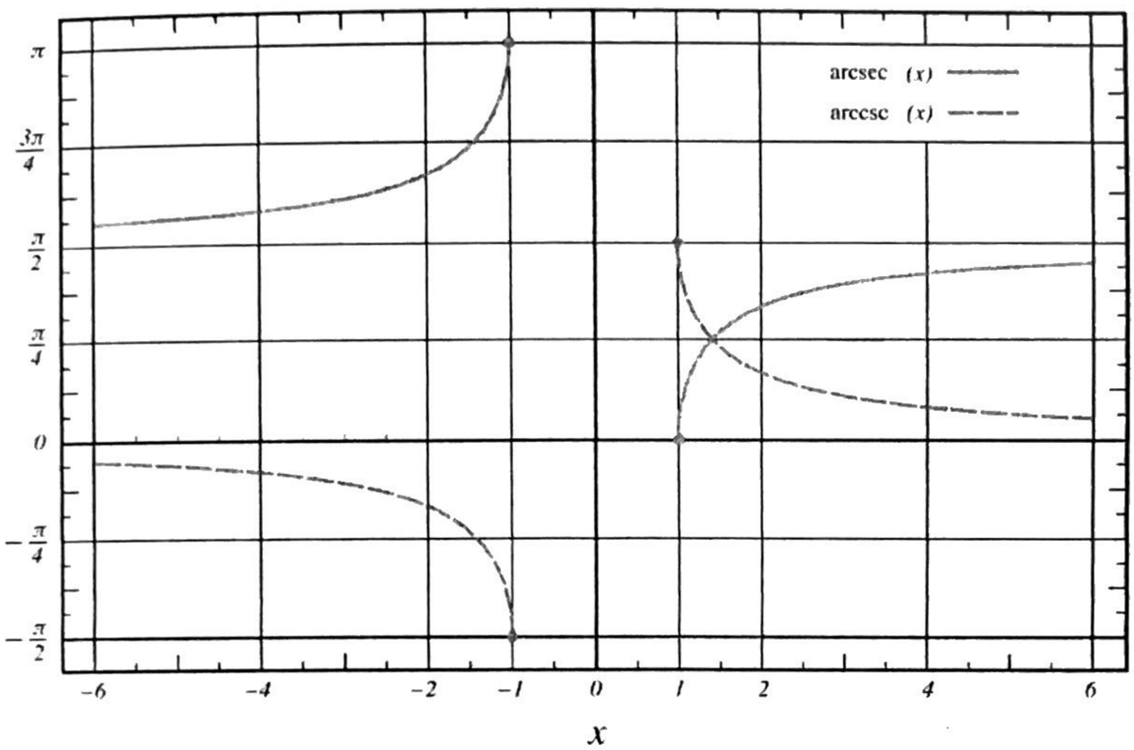


ii) $y = \text{Tan}^{-1}(x) \Rightarrow -\pi/2 < y \le \pi/2$



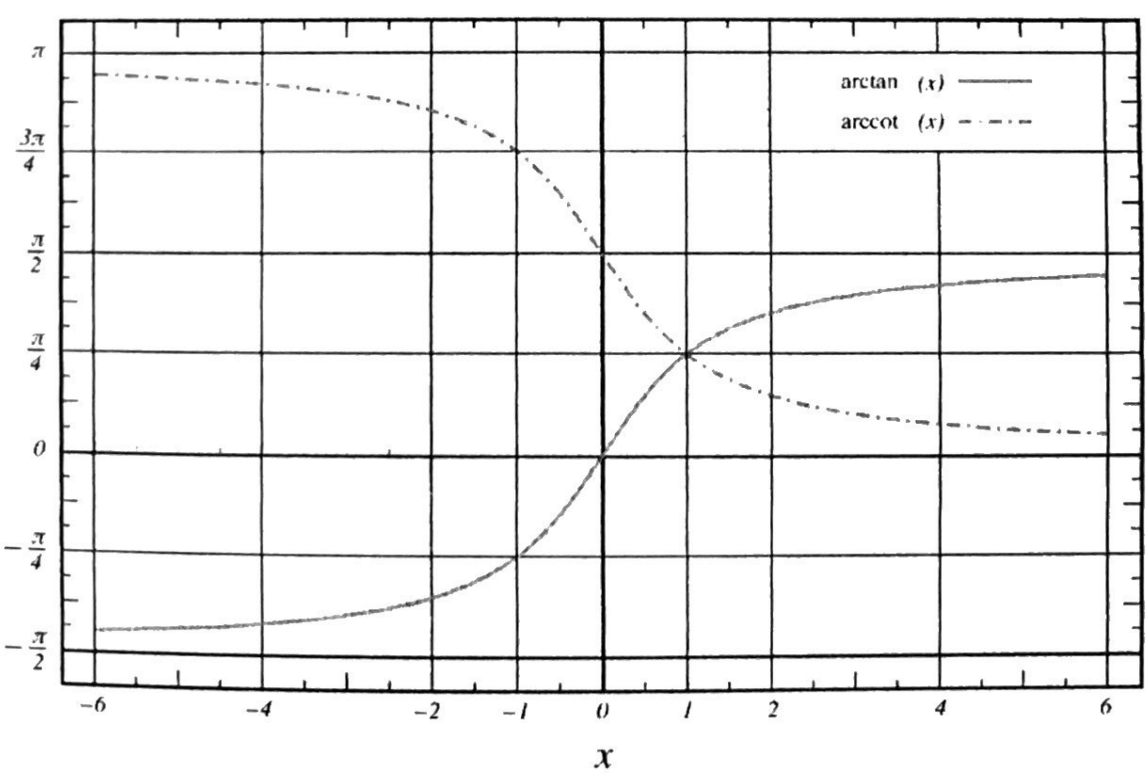
iii) & iv) $y = \text{cosec}^{-1}(x)$ and $y = \sec^{-1}(x)$. Picture taken from Wikipedia

http //upload.wikimedia.org/wikipedia/commons/3/32/ArcSec_and_ArcCsc.svg 24/08/2008 23 10



v) $y = \cot^{-1}(x)$. Picture taken from Wikipedia

ıttp //upload wikimedia org/wikipedia/commons/8/83/ArcTan_and_ArcCot svg 24/08/2008 23 10



2. i) $\mathrm{Sin}^{-1}\left(\frac{\sqrt{3}}{2}\right) = \pi/3 = 60\,\text{degrees}$

ii) $\mathrm{Sin}^{-1}\left(\frac{1}{\sqrt{2}}\right) = \pi/4 = 45\,\text{degrees}$

iii) $\mathrm{Sin}^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\pi/4 = -45\,\text{degrees}$

iv) $\sin\left\{\mathrm{Cos}^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right\} = 1/2$

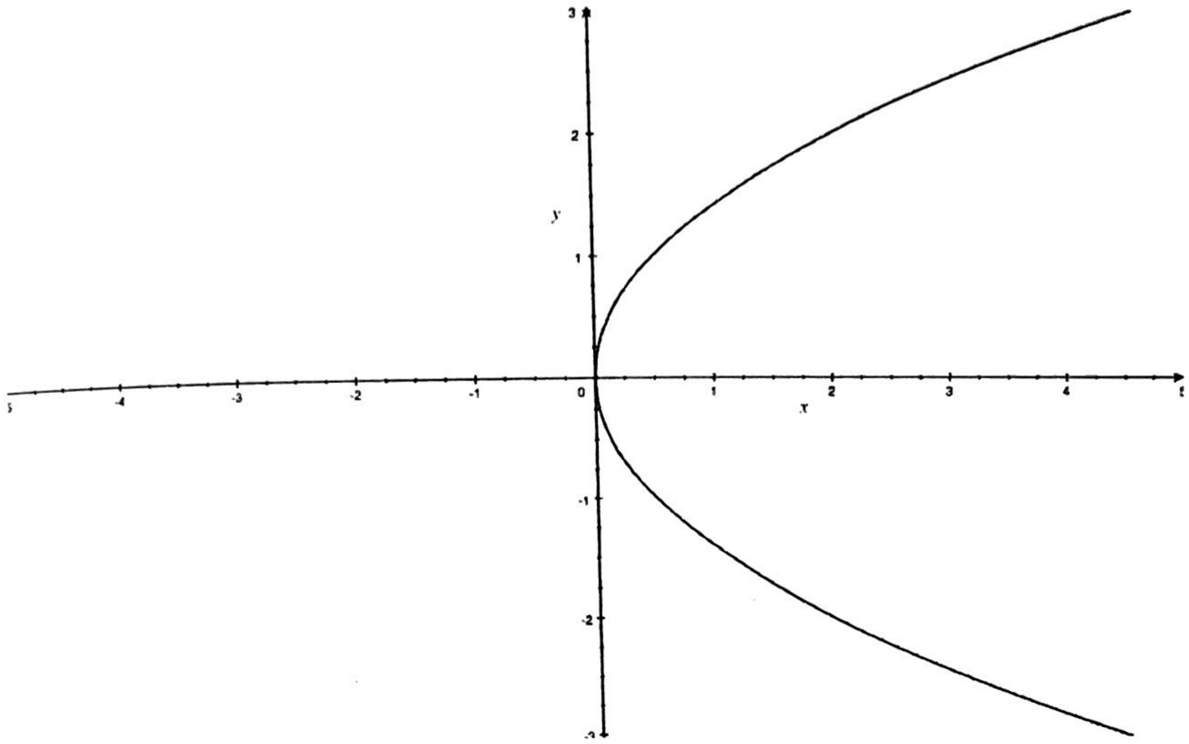
v) $\mathrm{Cos}^{-1}\left(\frac{1}{2}\right) + \mathrm{Cos}^{-1}\left(-\frac{1}{2}\right) = \pi$

vi) $\mathrm{Sin}^{-1}\left(\frac{1}{2}\right) + \mathrm{Sin}^{-1}\left(-\frac{1}{2}\right) = 0$

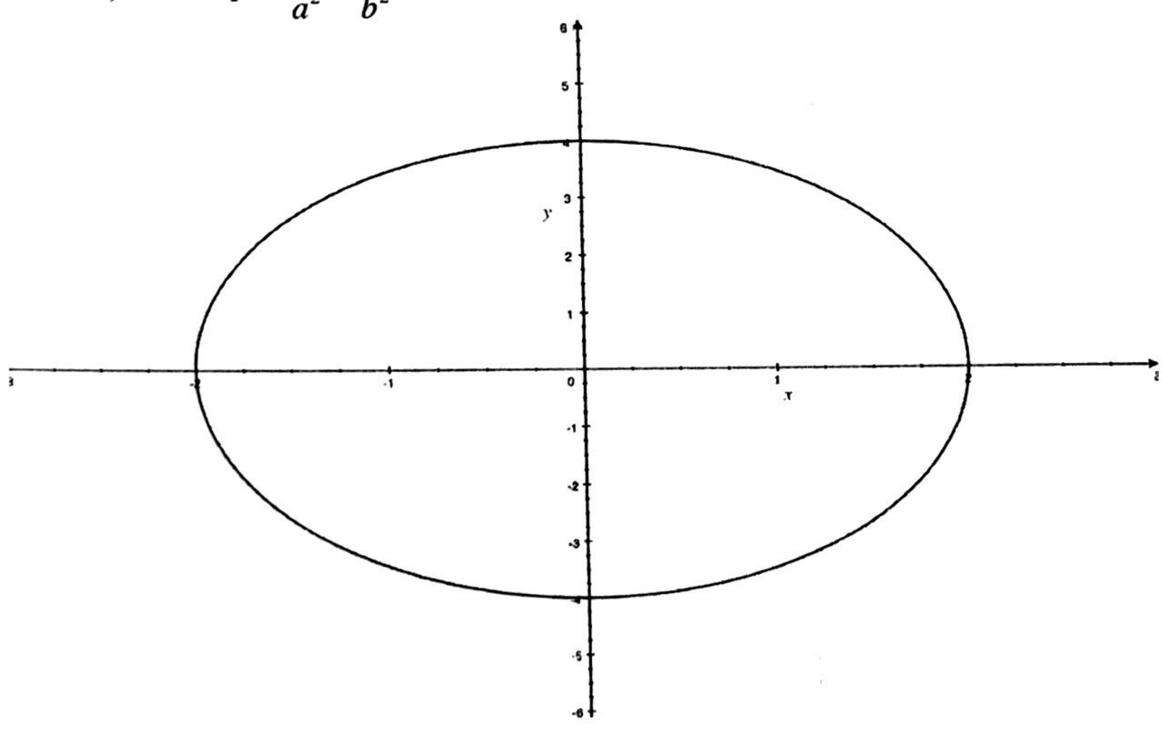
Exercise 7.1 Solutions

1. Plots

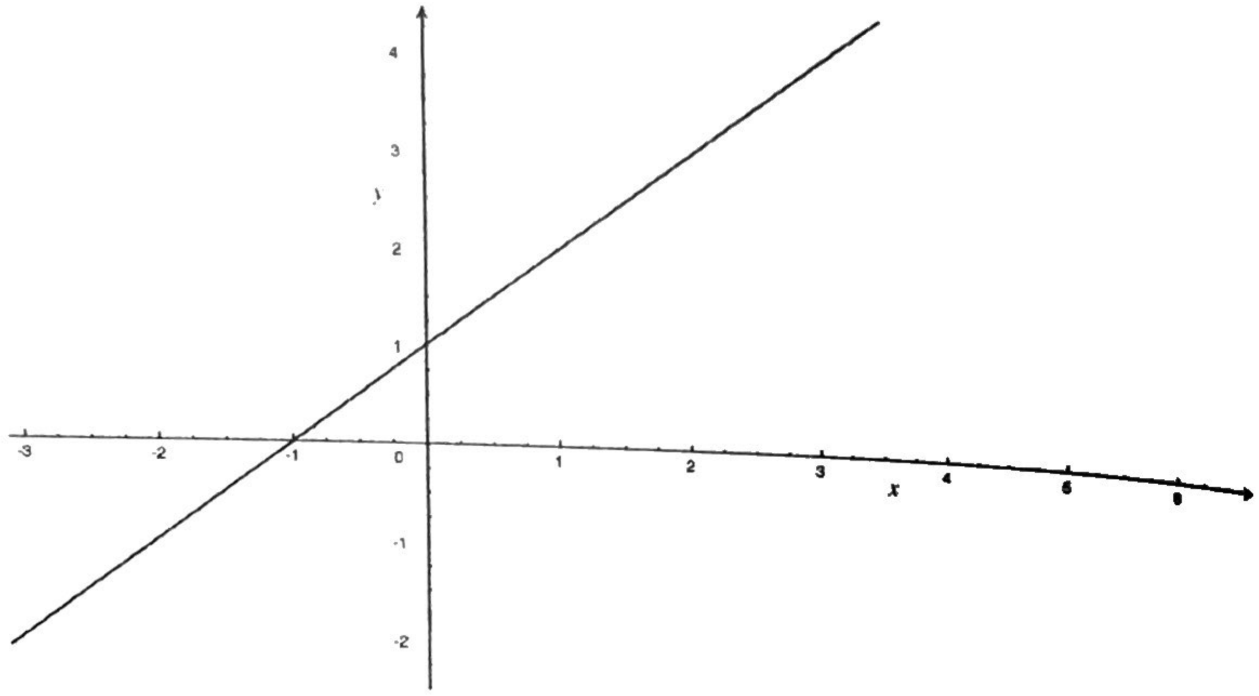
i) parabola $y^2 = 4ax$. Here $a = 1/2$



ii) ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Here with $a = 2$ and $a = 4$

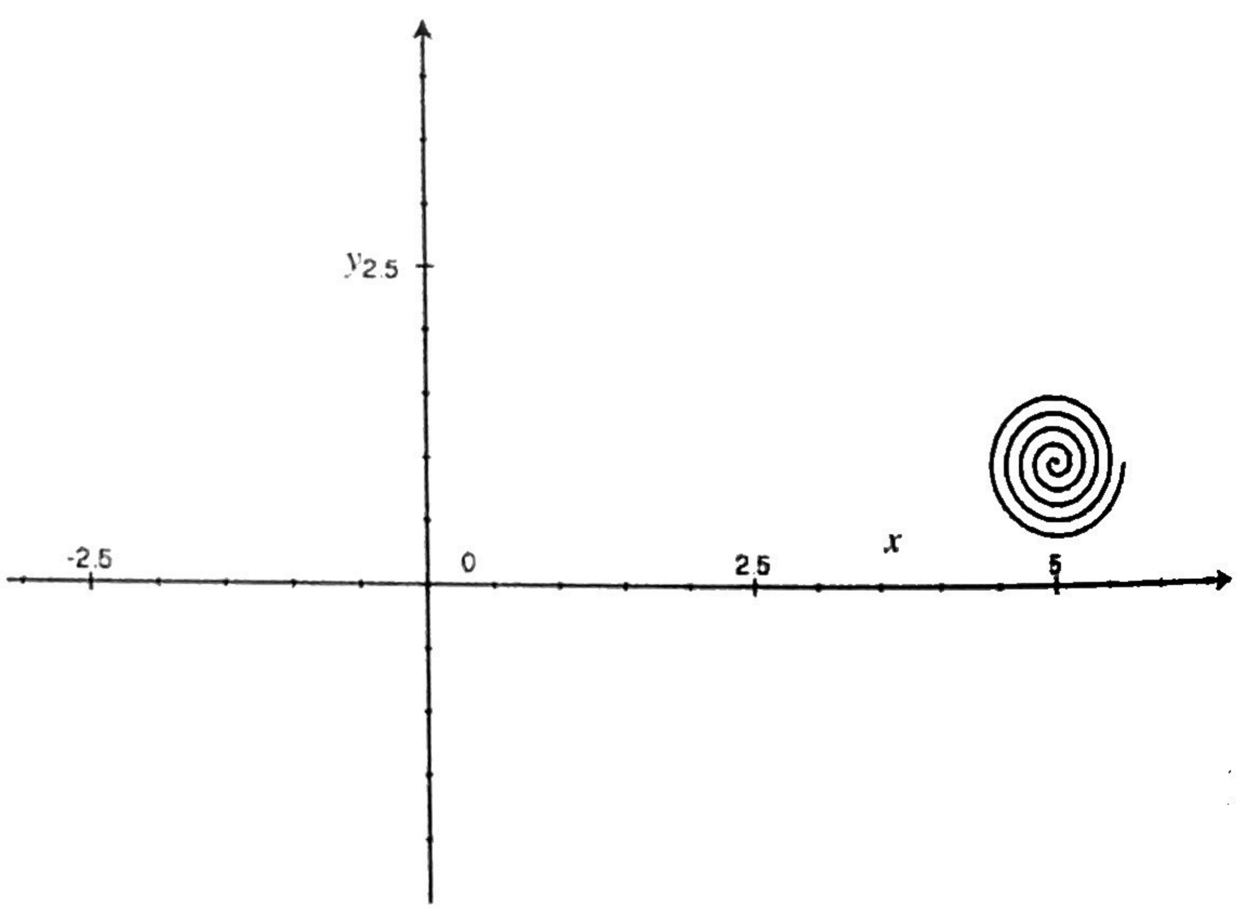


iii) straight line $y = x + 1$



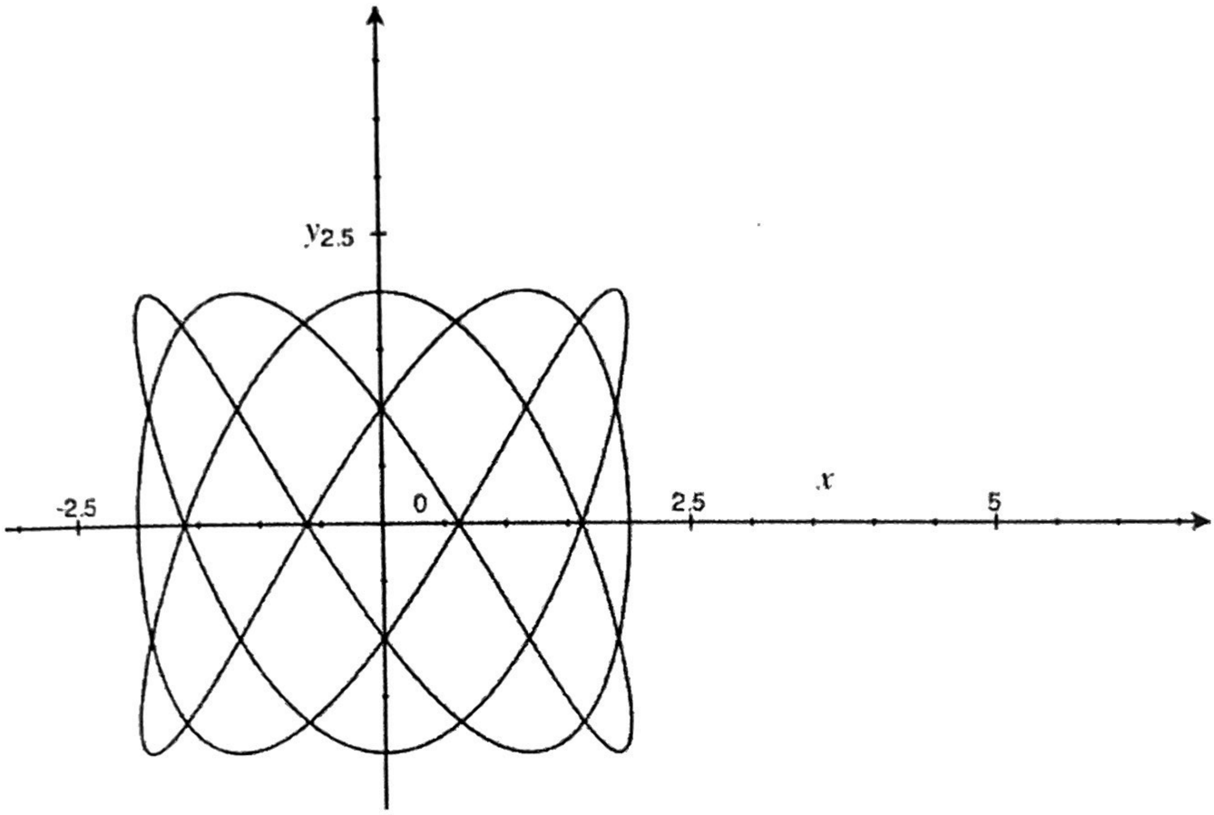
iv) Taken from the Mac Grapher examples

$$x = 5 + \frac{\theta}{50}\cos(\theta) \quad y = 1 + \frac{\theta}{50}\sin(\theta) \quad \theta = [0, 10\pi]$$



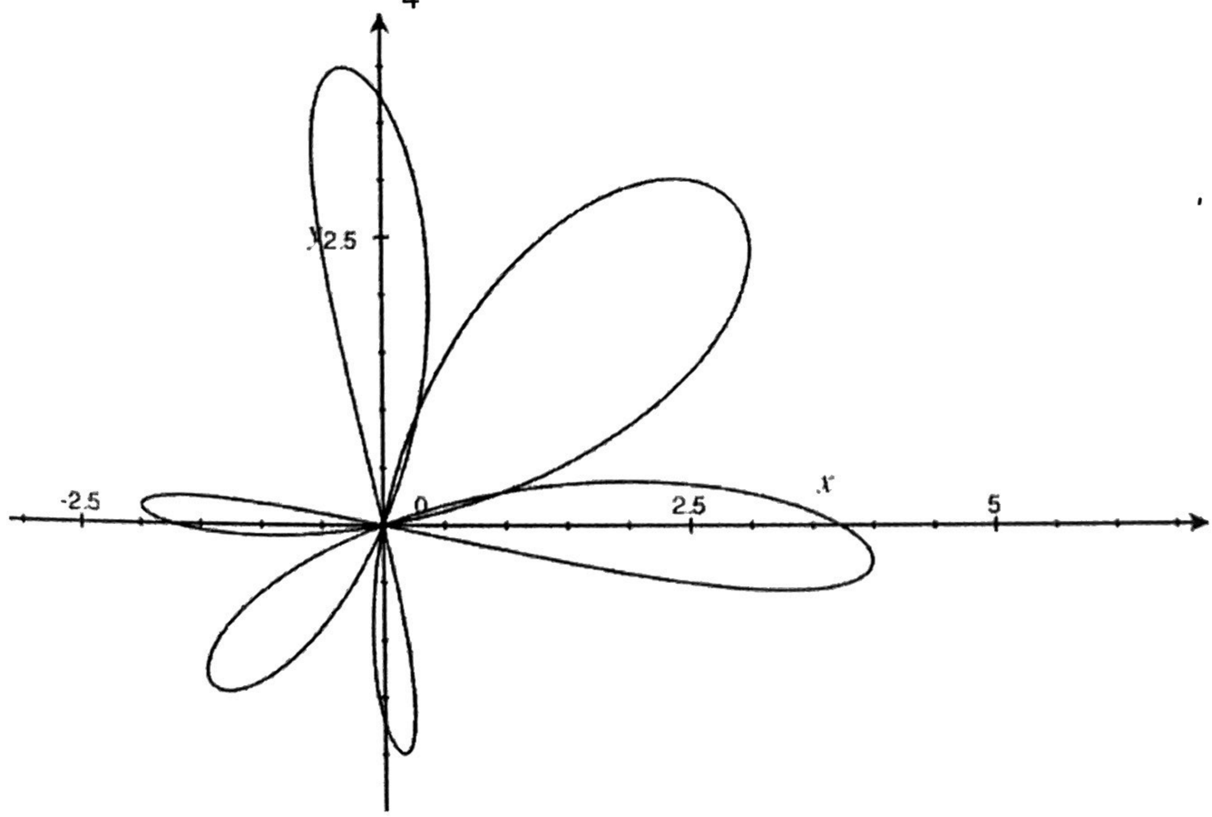
v) Taken from the Mac Grapher examples

$x = 2\cos(3t) \quad y = 2\sin(5t) \quad t = [0, 2\pi]$



vi) Taken from the Mac Grapher examples

$r = 1 + 3\cos(3t) \quad \phi = \dfrac{\pi}{4} - \sin(t) \quad t = [0, 2\pi]$



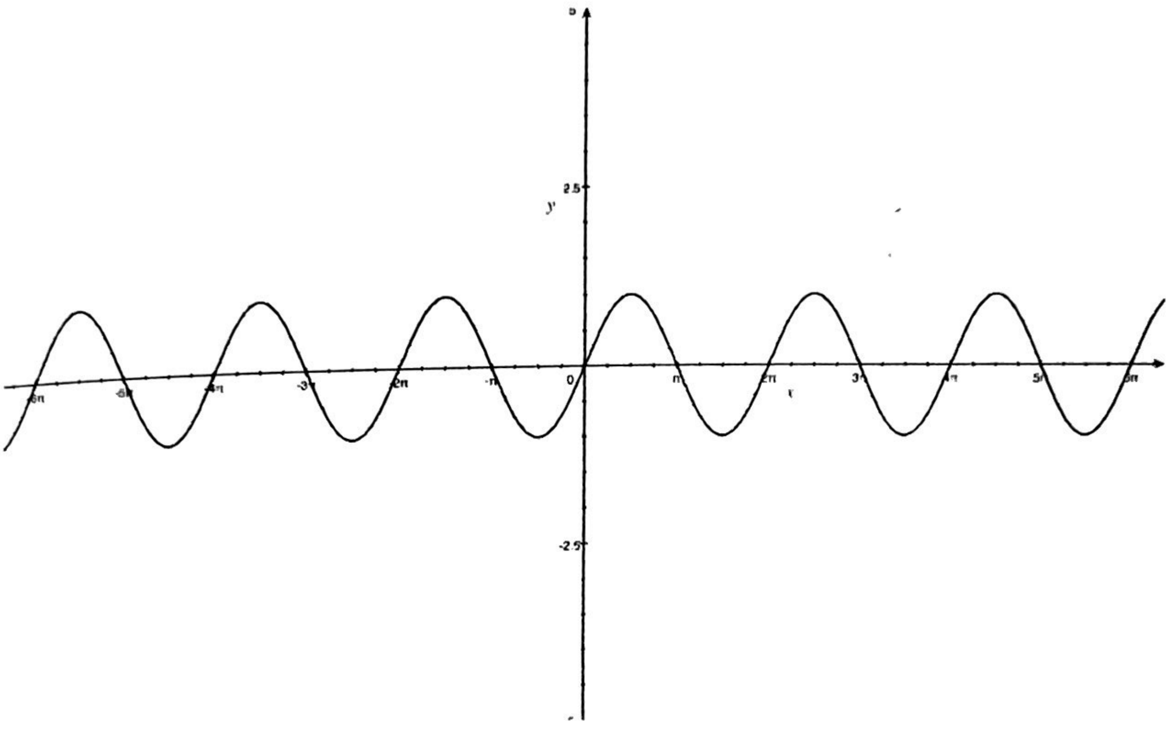
2. Let A be the point (a,α), and a pont P on the line be (r,θ). We must have $r\cos(\theta-\alpha)=a$. Using the usual parametric representations we get $\cos\alpha\, x+\sin\alpha\, y=a$, as the equation of the line.

3. The algebraic equation which is satisfied by the number $x=\sqrt[3]{2}+3\sqrt{2}$ is given by $x^6-54x^4-4x^3+972x^2-216x-5828=0$

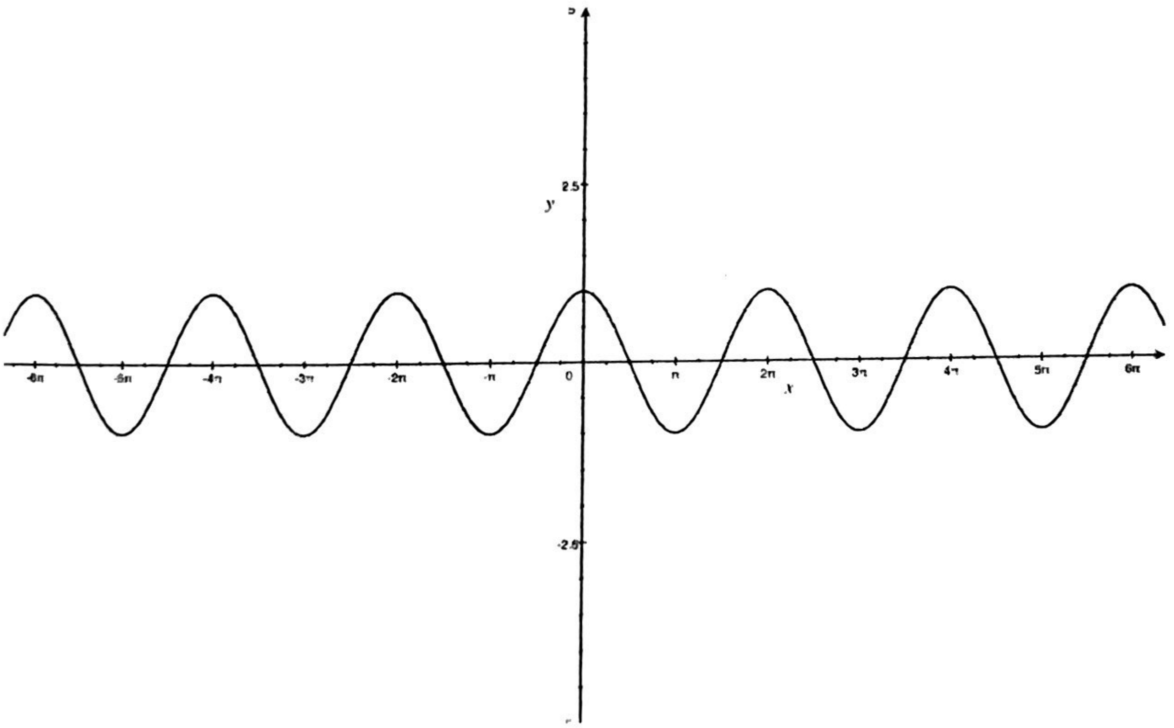
Exercise 8.1 Solutions

Recall the graphs for sine and cosine which bot have period 2π

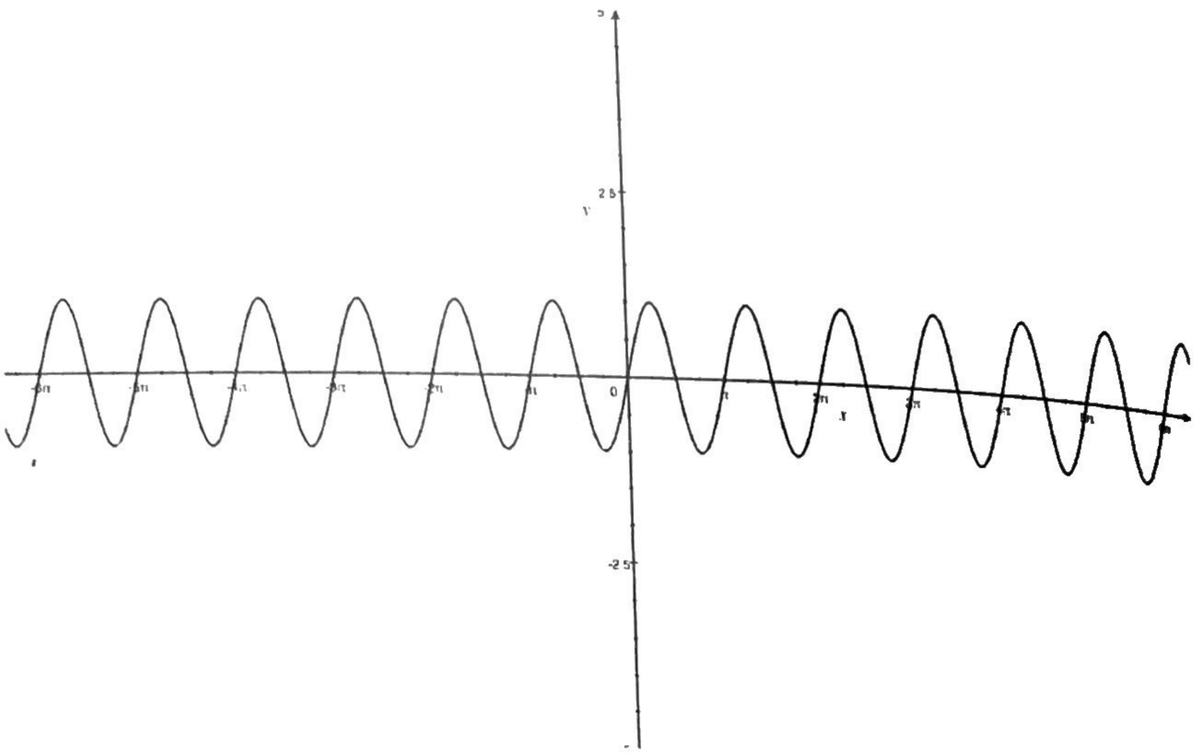
$y = \sin(x)$



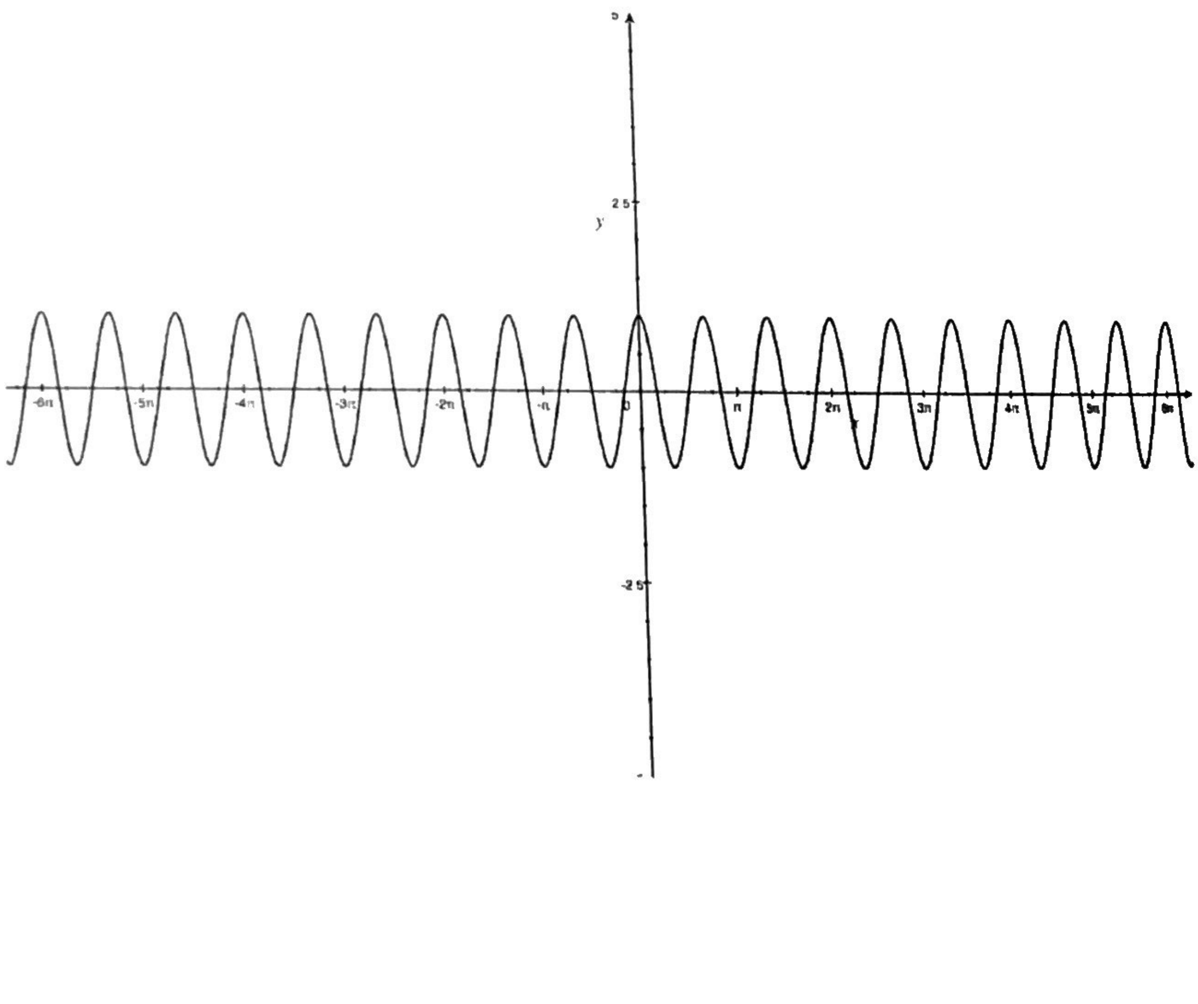
$y = \cos(x)$



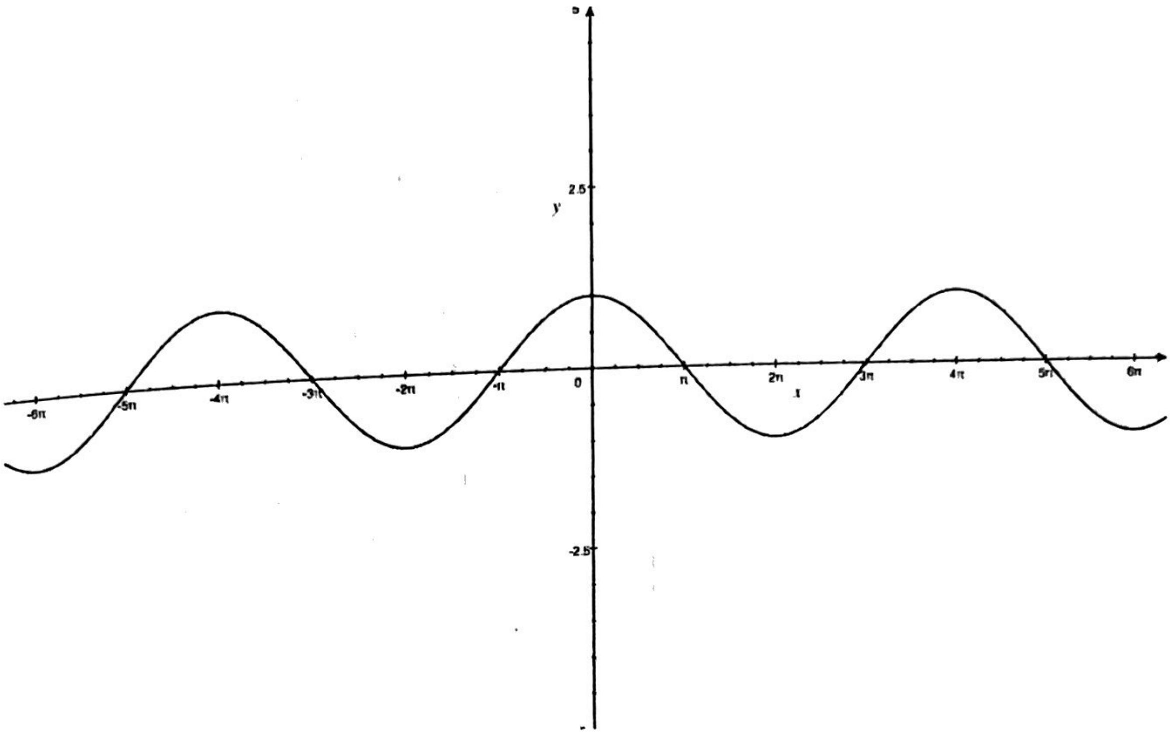
2. i) $y = \sin(2x)$ has period π



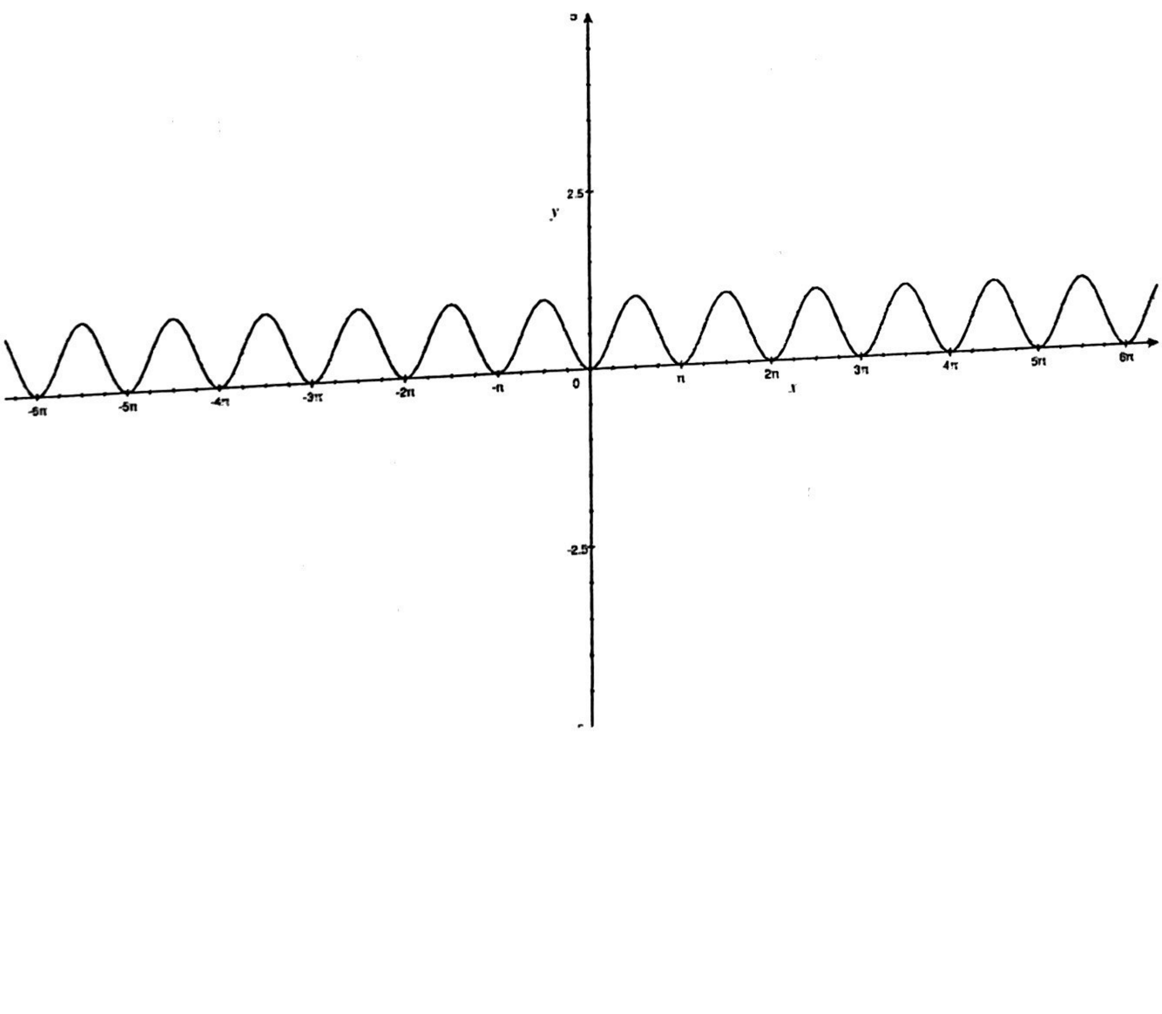
ii) $y = \cos(3x)$ has period $2\pi/3$



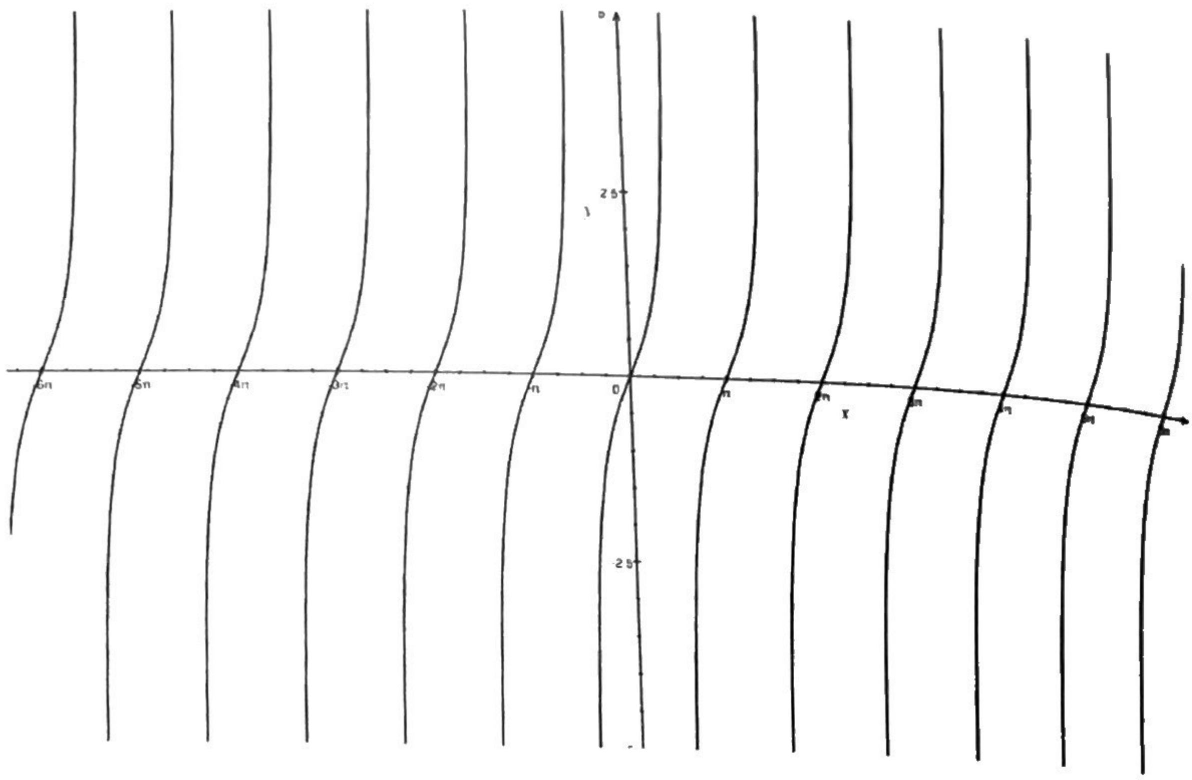
iii) $y = \cos\left(\dfrac{x}{2}\right)$ has period 4π



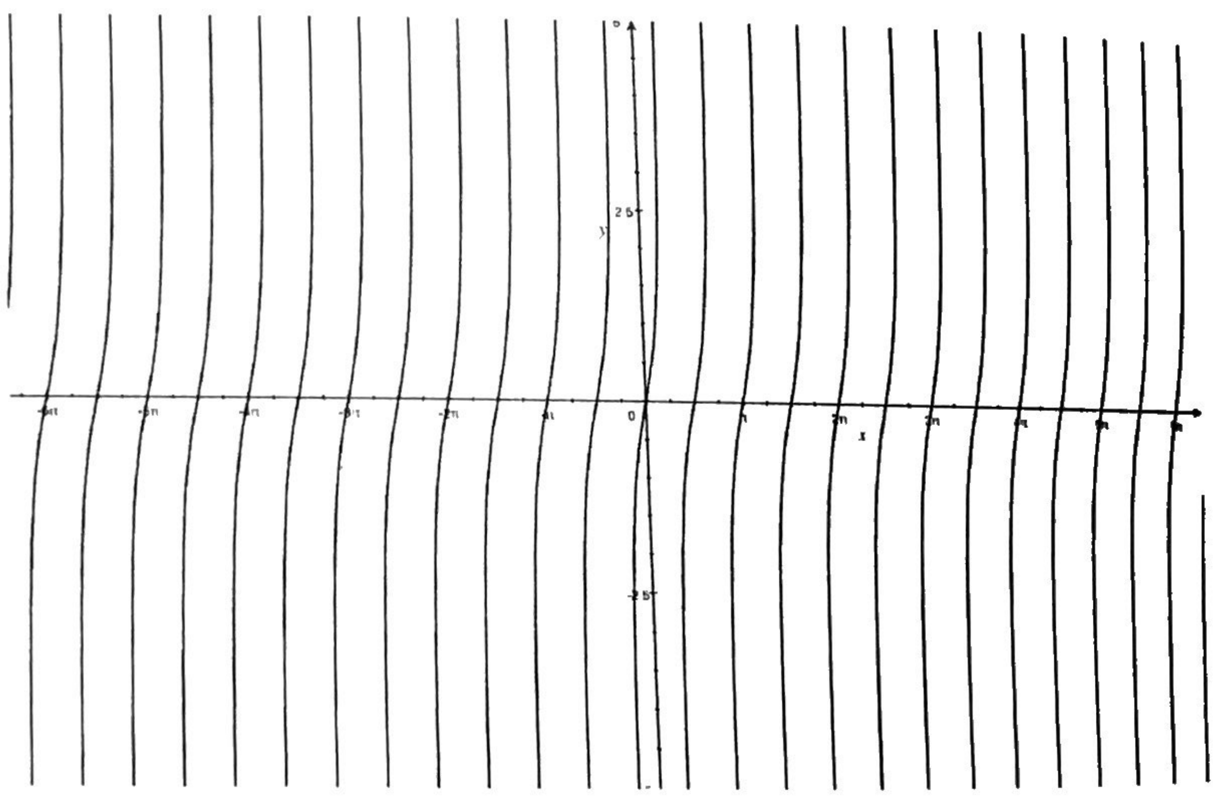
iv) $y = \sin^2(x)$ has period π



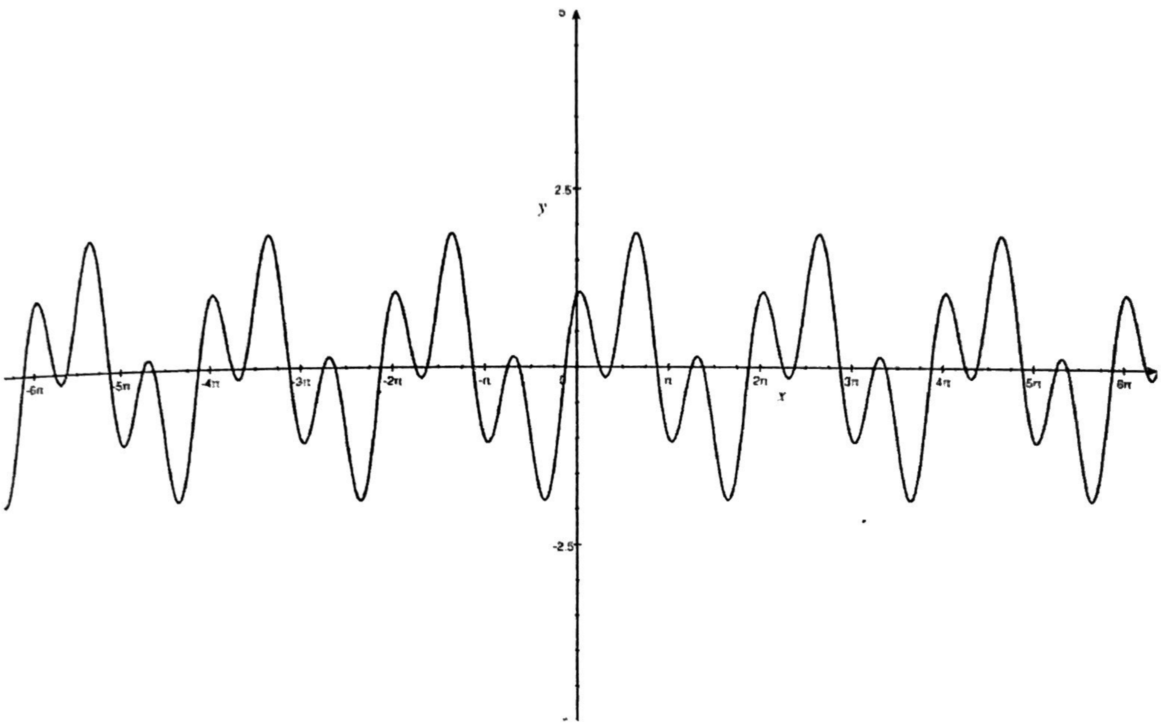
v) $y = \tan(x)$ has period π



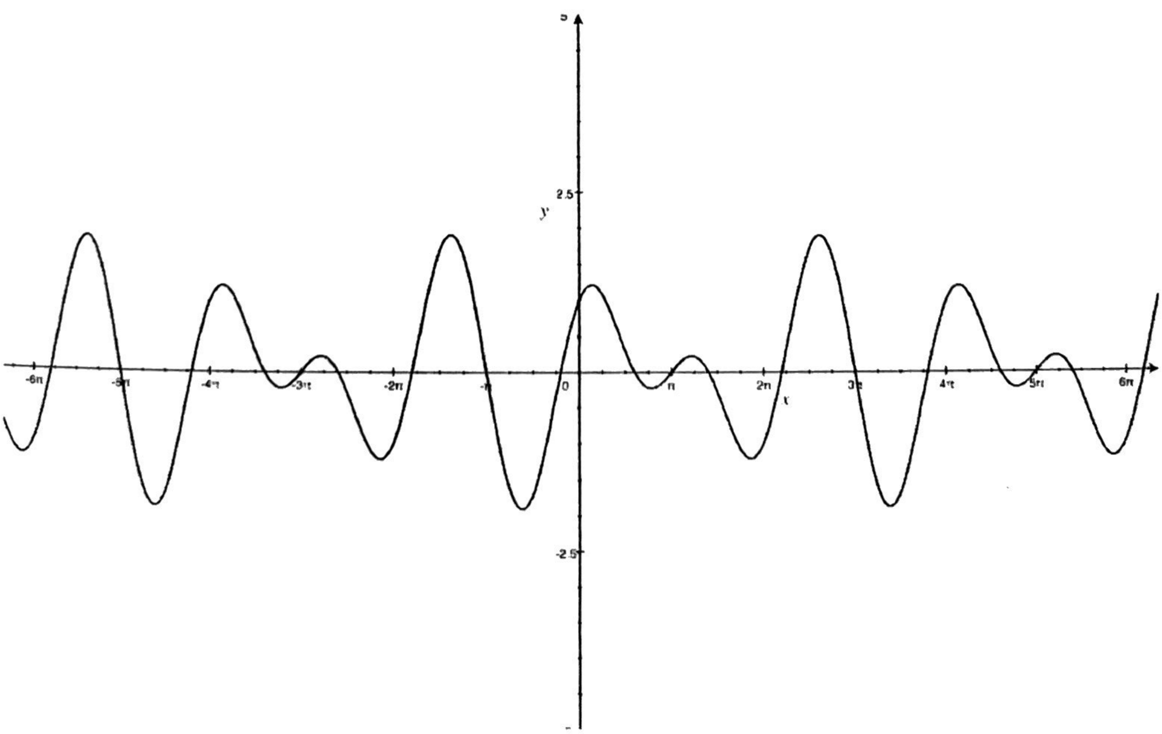
vi) $y = \tan(2x)$ has period $\pi/2$



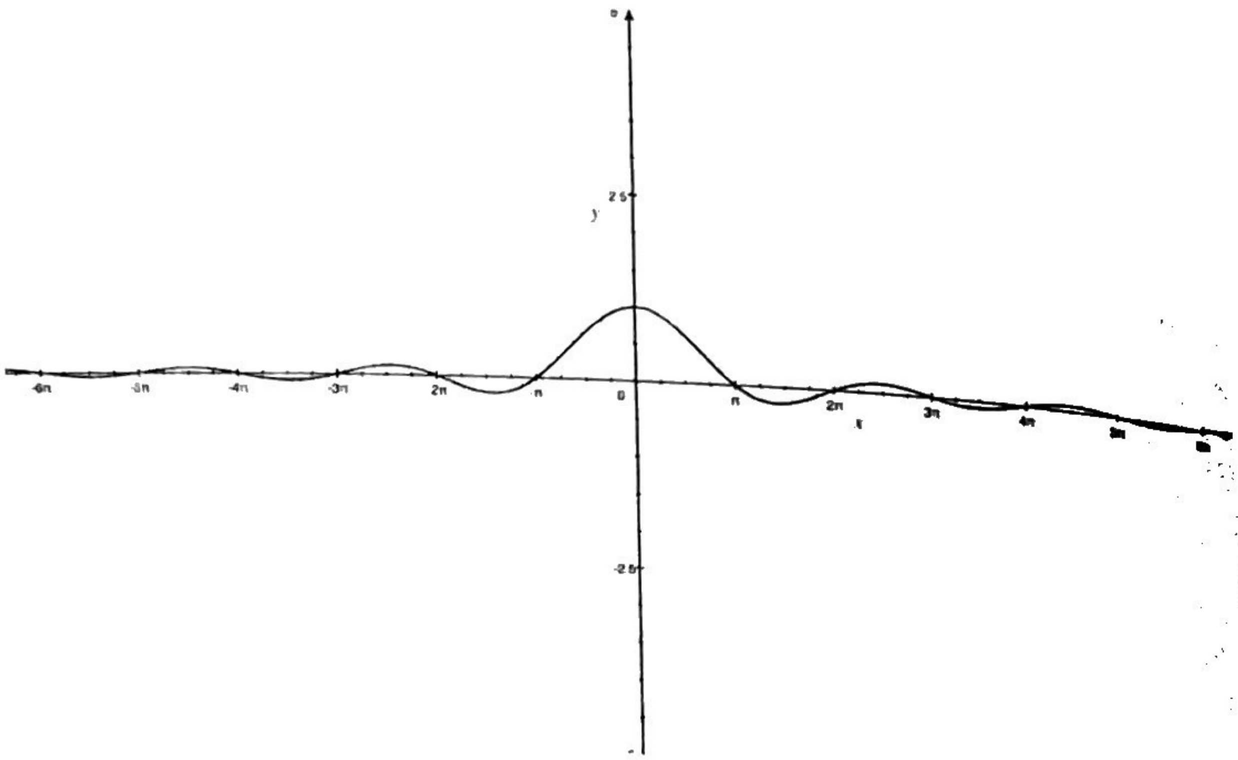
vii) $y = \sin(x) + \cos(3x)$ has period 2π



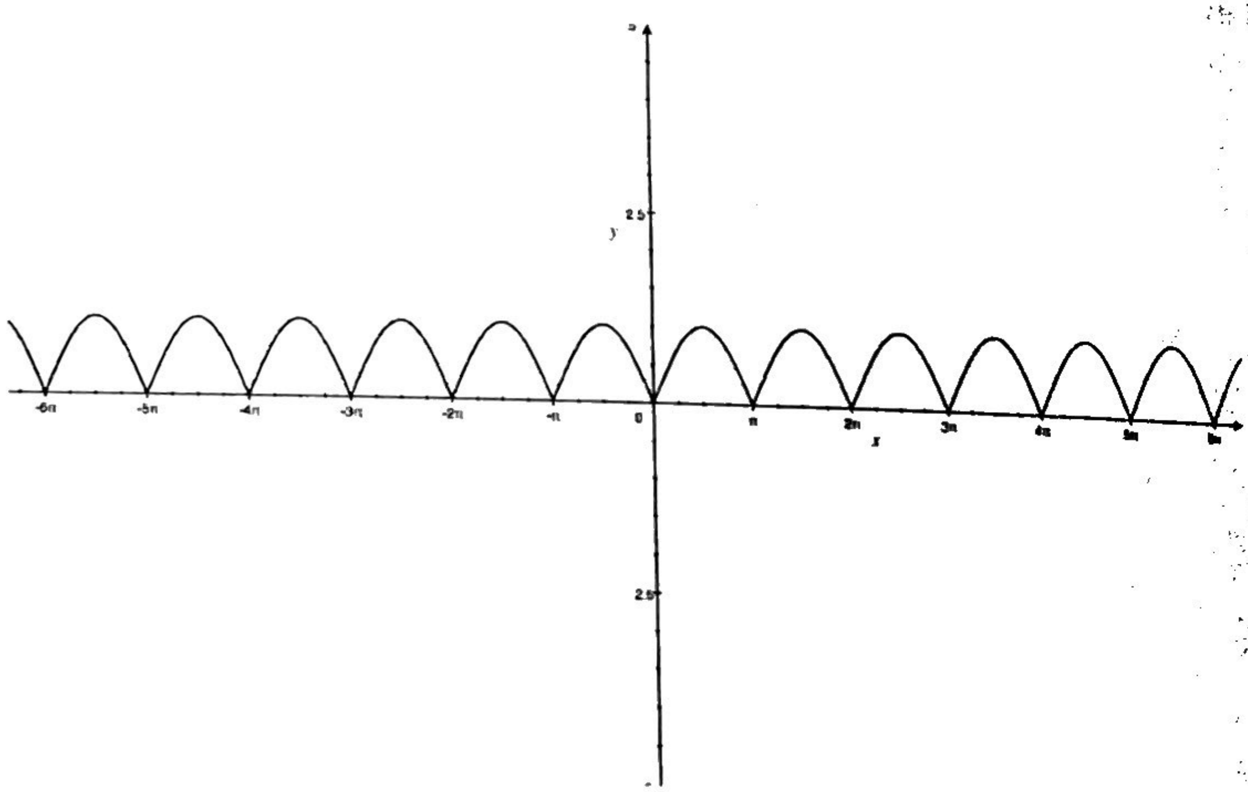
viii) $y = \sin(x) + \cos\left(\dfrac{3x}{2}\right)$ has period 4π



ix) $y = \dfrac{\sin x}{x}$ oscillates but the amplitude decays and so is not periodic



x) $y = |\sin(x)|$ has period π



Exercise 8.2 Solutions

1. Use the definitions f even $\Rightarrow f(-x) = f(x)$ and g odd $\Rightarrow f(-x) = -f(x)$ then consider $h(x) = f(x) \times g(x)$ etc.

2. i) $\sin(2x)$ is odd

   ii) $\sin(x^2)$ is even

   iii) $\sin^2(x)$ is even

   iv) $\tan\left(x/2\right)$ is odd

   v) $\dfrac{x^3 + 2x}{x^2 - 1}$ is odd

   vi) $\dfrac{x^2 - 2}{x^3 + 1}$ is neither even nor even

   vii) e^{x^2} is even

   viii) $\sin\left(x - \pi/2\right)$ is even

   ix) $x^3 + 6x$ is odd

   x) $x^2 + 2\sin(x)$ is neither even nor even

   xi) $e^{x^2}\cos(3x)$ is even

   xii) $\dfrac{1}{x}$ is odd

   xiii) $\dfrac{x}{\sqrt{(1 - x^2)}}$ is odd

   xiv) $x\sin(x)$ is even

   xv) $x\cos(x)$ is odd

   xvi) $\mathrm{Tan}^{-1}(x)$ is odd

Exercise 8.3 Solutions

1. i) $f(x) = x^2 + 2\sin(x)$ is neither odd nor even.

   Let $g(x) = \frac{1}{2}\{f(x) + f(-x)\} = x^2$ and $h(x) = \frac{1}{2}\{f(x) - f(-x)\} = 2\sin(x)$,

   then $f(x) = g(x) + h(x)$

   ii) $f(x) = \dfrac{x^2 - 2}{x^3 + 1}$ is neither odd nor even

   Let $g(x) = \frac{1}{2}\{f(x) + f(-x)\} = \dfrac{x^2 - 2}{1 - x^6}$ and

   and $h(x) = \frac{1}{2}\{f(x) - f(-x)\} = \dfrac{x^3(x^2 - 2)}{x^6 - 1}$,

   then $f(x) = g(x) + h(x)$

   iii) $f(x) = x^2(1 + x + 2^x)$ is neither odd nor even

   Let $g(x) = \frac{1}{2}\{x^2(2 + 2^x + 2^{-x})\}$ and $h(x) = \frac{1}{2}\{x^2(2x + 2^x - 2^{-x})\}$,

   then $f(x) = g(x) + h(x)$

2. Own function.

3. Beware. Usually defining $g(x) = f(x) + f(-x)$ leads to the function $g(x)$ being even but in this case the domains are not compatible.

Exercise 8.4 Solutions

1. i) $y(x) = 3x^2 + x + 2$ is continuous everywhere

ii) $f(x) = \dfrac{x^3 + 4x + 6}{x^2 - 6x + 8}$ is not defined at $x = 2$ and $x = 4$

iii) $f(x) = \sec\left(\dfrac{x}{2}\right)$ not defined when

$x = (2n + 1)\pi, \quad n = 0, \pm 1, \pm 2, \ldots.$

iv) $y(x) = |2x| - |1 - x|$ is continuous everywhere

v) $y(x) = \sin\left(\dfrac{1}{x}\right)$ not defined when $x = 0$

vi) $y(x) = \tan\left(\dfrac{1}{x}\right)$ not defined when

$x = \dfrac{2}{(2n + 1)\pi}, \quad n = 0, \pm 1, \pm 2, \ldots.$

vii) The function

$$f(x) = \begin{cases} 1 & x \text{ an integer} \\ 0 & x \text{ not an integer} \end{cases}$$

is discontinuous at all integers.

Exercise 8.5 Solutions

1. i) $\frac{3x+1}{4x+3} \to \frac{1}{3}$ as $x \to 0$

ii) To do this limit write the top as the difference of two squares since in this course we have not covered differenciation as yet hence L'Hopitals Rule has not been covered. $\frac{x^2-9}{x-3} \to 6$ as $x \to 3$

iii) $\lim_{x\to 0}|x| = 0$

iv) $\lim_{x\to 0}\frac{|x|}{x}$ has no limit since $x > 0$ $\frac{|x|}{x} \to 1$ but $x < 0$ $\frac{|x|}{x} \to -1$ and remember that the two limits should be the same as the value of the function at $x = 0$

v) $\lim_{x\to 1}[x]$ has no limit since $x < 1$ $[x] = 0$ but $x > 1$ $[x] = 1$ same comment as above

vi) $\lim_{x\to 1/2}[x] = 0$

vii) $\lim_{x\to 0}\frac{(2+x)^3-8}{x} = 12$

viii) $\lim_{x\to 1/3}\frac{3x^3-x^2-3x+1}{6x^2+13x-5} = \frac{-8}{51}$

2. i) $\frac{x+2}{x^2+x+1} \to 0$ as $x \to \infty$

ii) $\frac{x^2+6x+4}{3x^2+7x+2} \to \frac{1}{3}$ as $x \to \infty$

iii) $\left\{\sqrt{x+1}-\sqrt{x}\right\} \to 0$ as $x \to \infty$

iv) $\sqrt{x^2+x+1}-x$ looks at first like it should converge to 0 as $x \to \infty$, but does it? In fact it converges to 1/2

3. i) $\frac{\sin(x)}{x} \to 0$ as $x \to \pm\infty$, but the sign alternates

ii) $\frac{\cos(x)}{x} \to 0$ as $x \to \pm\infty$

Exercise 8.6 Solutions

1. i) $\lim_{x \to 0} \sin(x) = 0$

   ii) $\lim_{x \to 0} \cos(x) = 1$

   iii) $\lim_{x \to 0} \frac{1 - \cos(2x)}{x} = 0$

2. $\lim_{x \to a} \frac{\sin(x-a)}{x^2 - a^2} = \frac{1}{2a}$

3. As $x \to 0$ the sine term oscillates but $x^2 \to 0$. However the function is not defined for $x = 0$ therefore the limit does not exist.

4. $\lim_{x \to a} \frac{1}{x}\left\{\frac{1}{a+x} - \frac{1}{a}\right\} = -\frac{1}{a^2}$, $a \neq 0$. The limit is not defined, i.e. does not exist for $a = 0$.

Exercise 9.1 Solutions

1. First show that the square of an odd number when divided by 4 and 8 both leave a remainder 1 by using the method of proof by induction. Then assume that rational solutions exist and go to prove by contradiction that this assumption is not possible.

2. i) Use formula $x^2 + 11 = 7x \quad \Rightarrow x = \frac{7+\sqrt{5}}{2}, \frac{7-\sqrt{5}}{2}$ i.e. 4.618, 2.382

ii) Factorizes $x^2 - 4x = 5 \quad \Rightarrow x = -1, 5$

iii) Factorizes $x^2 - 5x + 4 = 0 \quad \Rightarrow x = 1, 4$

iv) Factorizes $y^2 + 2y + 1 = 0 \quad \Rightarrow y = -1, -1$

v) Factorizes or use the formula $2y^2 + 4y = 5/2 \quad \Rightarrow y = 1/2, -5/2$

vi) Use formula $x^2 + 9x - 3 = 0 \quad \Rightarrow x \approx 0.322, 9.322$

vii) Factorizes or use the formula $2x^2 + 5x = 12 \quad \Rightarrow x = 3/2, -4$

viii) Factorizes or use the formula $8x^2 - 2x - 3 = 0 \quad \Rightarrow x = 3/4, -1/2$

ix) Use formula $5x^2 - 17x + 10 = 0 \quad \Rightarrow x = 2.64, 0.76$

x) Use formula $4x^2 - 8x + k = 04x^2 - 8x + k = 0 \quad \Rightarrow x = 1 \pm \frac{1}{2}\sqrt{4-k}$. So If if $k = 4$ then there are two equal roots of 1

3. You can use the formula or multiply out $a(x-\alpha)(x-\beta)$ and compare coeffcents of powers of x.

4. Use result of Question 2 to give $q^2x^2 - xp(p^2 - 3q) + q = 0$.

5. i) Factorizes $x^2 - 3x + 2 = 0 \quad \Rightarrow x = 1, 2$

ii) No real roots $2x^2 + x + 5 = 0$, and we have not covered complex numbers yet, so leave for now – although I am sure that you can do it.

iii) Factorizes $3x^2 - 17x + 10 = 0 \quad \Rightarrow x = 2/3, 5$

iv) Factorizes $5x^2 - 7x = 0$

v) Factorizes with equal roots $x^2 + 4x + 4 = 0 \quad \Rightarrow x = -2$, twice

vi) Use the formula $4x^2 + 4x - 1 = 0 \quad \Rightarrow x = 0.207, -1.207$

vii) No real roots $x^2 + 2x + 2 = 0$, so again leave for now

viii) Use the formula $x^2 - 2x = 2 \quad \Rightarrow x \approx 2.732, -0.732$

6. i) Let $y = 2^x$ then $2^{2x} - 3(2^x) + 2 = 0 \Rightarrow x = 1, 0$

ii) Use formula $p^2 = -(7p + 8) \quad \Rightarrow p = -1.438, -5.562$

iii) Use formula $x + \dfrac{1}{3x} = 2 \quad \Rightarrow x = 1.816, 0.184$

iv) Square both sides then the equation factorizes

$\sqrt{10 - 3x} = x \quad \Rightarrow x = 2, -5$

v) Square and square again after rearranging $\sqrt{x+5} = 4 - \sqrt{x-3} \quad \Rightarrow x = 4$

vi) Let $y = x^{1/4}$ $\quad 2x^{1/2} - 15x^{1/4} + 27 = 0 \quad \Rightarrow x = 81, 410.0625$

vii) Let $y = x^2$ $\quad x^4 - 3x^2 + 2 = 0 \quad \Rightarrow x = \pm 1, \pm\sqrt{2}$

viii) $\dfrac{x^2}{(6-x)^2} = 4 \quad \Rightarrow x = 12, 4$

ix) Use formula $z - 1.0 \times 10^{-7} z - 1.0 \times 10^{-14} = 0$

$\Rightarrow z \approx 1.618 \times 10^{-7}, \quad -0.618 \times 10^{-7}$

7. Use Pythagarus's theorem. Either $x = 12$ or $x = -16$ but only a positive value is meaningful here.

8. Use (flow rate) x (time) = volume. Leads to a quadratic and then $t = 45$ minutes or $t = 30$ minutes.

9. These are not linear equations but luckily we can still solve them.

i) $x = 1, \quad y = 1, \quad z = 0$

ii) Either $\begin{Bmatrix} x = 0 \\ y = 4 \end{Bmatrix}$ or $\begin{Bmatrix} x = 4 \\ y = 0 \end{Bmatrix}$

iii) Either $\begin{Bmatrix} x = 9/2 \\ y = 9/2 \end{Bmatrix}$ or $\begin{Bmatrix} x = 6 \\ y = 3 \end{Bmatrix}$

iv) 5 x equation (1) + equation (2) eliminates constant term.

$$\Rightarrow x = \frac{3}{5}y \quad \text{or} \quad x = \frac{y}{2}$$

substitution back into equation (1) leads to either $\begin{Bmatrix} x = \pm 3 \\ y = \pm 5 \end{Bmatrix}$ or $\begin{Bmatrix} x = \pm 1 \\ y = \pm 2 \end{Bmatrix}$

v) Similar to part iii) take 11 x equation (1) - equation (2) to

eliminate the constant $\Rightarrow m = \frac{3}{7}n \quad \text{or} \quad m = \frac{n}{2}$

So either $\begin{Bmatrix} n = \pm 7 \\ m = \pm 3 \end{Bmatrix}$ or $\begin{Bmatrix} n = \pm 2 \\ m = \pm 1 \end{Bmatrix}$

Exercise 9.2 Solutions

1. i) $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + ...$

   ii) $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + ...$

2. i) $q(x) = 2x^2 + x - 1$

   ii) $q(x) = x^5 - 7x^2 + 2$

   iii) $q(x) = x^4 + 3$

   iv) $q(x) = x^4 + x^3 + x^2 + x + 1$

   v) $q(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$

3. The answer is 5 in both cases. Long division was not easy here. Why is this? If this is too long try the simpler version $x + x^3 + x^9$.

Exercise 9.3 Solutions

1. i) $x^2 - y^2 = (x - y)(x + y)$

ii) $x^2 + 2xy + y^2 = (x + y)^2$

iii) $x^2 - 2xy + y^2 = (x - y)^2$

iv) $acx^2 + adxy + bcxy + bdy^2 = (ax + by)(cx + dy)$

v) $x^2 - 4x - 5 = (x + 1)(x - 5)$

vi) $x^2 + x = x(x + 1)$

vii) $x^2 - 1 = (x - 1)(x + 1)$

viii) $2x - x^2 + 3 = (x + 1)(3 - x)$

ix) $x^2 + x - 2 = (x - 1)(x + 2)$

x) $x^2 - 5x + 6 = (x - 2)(x - 3)$

xi) $x^3 - 2x^2 + x = x(x - 1)^2$

xii) $x^3 - 3x + 2 = (x - 1)^2(x + 2)$

xiii) $x^3 + x^2 + 4x + 4 = (x + 1)(x^2 + 4)$

xiv) $(a + b + c)^3 - a^3 - b^3 - c^3 = 3(a + b)(b + c)(a + c)$

xv) $(a + b + c)$ is a factor. Find the other. Look back at Exercise 3.3 No. 7

2. i) $x^2 + 14x + 9 = 0 \quad \Rightarrow x = -7 \pm 2\sqrt{10}$

ii) $x^2 - 4x - 5 = 0 \quad \Rightarrow x = 5, -1$ also see Question 1 (v)

iii) $4x^2 + 4x - 5 = 0 \quad \Rightarrow x = \dfrac{-1 \pm \sqrt{6}}{2}$

iv) $x^2 - 6x + 2 = 0 \quad \Rightarrow x = 3 \pm \sqrt{7}$

v) $3x^2 - 6x - 1 = 0$. There are two ways you can think about this.

Either $3x^2 - 6x - 1 = 3\left(x^2 - 2x - \tfrac{1}{3}\right) = 3\left\{(x - 1)^2 - \tfrac{4}{3}\right\} \Rightarrow x = 1 \pm \dfrac{2}{\sqrt{3}}$,

or $3x^2 - 6x - 1 = \left(x\sqrt{3} - \sqrt{3}\right)^2 - 4 \Rightarrow x = \dfrac{\sqrt{3} \pm 2}{\sqrt{3}}$

3. Try the factors of 24, i.e. 2, 3, 4, 6, 8 and 12 to give

$$x^3 - 9x^2 + 26x - 24 = (x - 2)(x - 3)(x - 4).$$

4. Try $x = \pm 1$ and show that $x^5 + 1 = (x + 1)(x^4 - x^3 + x^2 - x + 1)$.

5. $a = 1$

6. Trial and error looking at the factors of the constant term and the highest order coefficient

   i) $x^3 - 4x + 2 = 0$ is cubic, so 3 roots. Roots bounded by (-3,-2), (0,1), and (1,2)

   ii) $x^3 - 8x^2 - 32x + 14 = 0$, cubic, so 3 roots, bounded by (-4,-3), (0,1), and (10,11)

   iii) $3x^4 + 4x^3 - 12x^2 + 1 = 0$, four roots this time bounded by (-3,-2), (-1,0), (0,1), and (1,2)

   iv) only 2 real roots of $x^4 - 2x - 1 = 0$ bounded by (-1,0), and (1,2)

7. $x^3 + 6x^2 + 11x + 6 = (x+1)(x+2)(x+3)$ and $x^2 + 3x + 2 = (x+1)(x+2)$ so easy to plug in values for $x = -1, -2, -3$ to show that

$$x^3 = a(x^3 + 6x^2 + 11x + 6) + b(x^2 + 3x + 2) + c(x+1) + d$$

when $a = 1, b = -6, c = 7\ d = -1$.

8. The answer is 5 in both cases. Long division was not easy here. Why is this? Here you are encouraged to assume that there is a factor plus a remainder $f(x) = (x-1)q(x) + r$ or $f(x) = (x^2 - 1)q(x) + rx + s$, note the need for the linear term in the remainder this time. Now try some trial values for *x*, just like you do for partial fractions.

Exercise 9.4 Solutions

1. Use $y = \frac{1}{x}$ $\Rightarrow a + by + cy^2 + dy^3 = 0$ has roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$.

2. $y^3 + 30y^2 + 600y - 3000 = 0$

3. Symmetric coefficients so if a α is a root so is $1/\alpha$. This leads to a quadratic factor. We can then use synthetic division to find a second quadratic factor. Then compare coefficients to find roots. $x = -2 \pm \sqrt{3}$ and $x = \frac{-7 \pm \sqrt{33}}{4}$, check $\alpha = 1/\alpha$. Or a useful trick for polynomials with symmetric coefficients is to divide first by the leading power, here x^2, and we can then re-arrange the equation to be a quadratic in terms of the new variable $y = x + \frac{1}{x}$.

4. The substitution leads to $y^4 - 13y^2 + 36 = 0$ now let $y^2 = b$ say, then
$$b^2 - 13b + 36 = 0 \quad \Rightarrow b = 9, 4.$$
Hence $y = \pm\sqrt{b} = \pm 3, \pm 2 \quad \Rightarrow x = 1, -3, 2, -4$.

5. α is a root so $\alpha^4 + \alpha^3 - 4\alpha^2 - 4\alpha + 1 = 0$, call this equation (1). Let $\beta = 2 - \alpha^2$, that is $\alpha = (2-\beta)^{1/2}$. Put this into equation 1 and show that β satisfies the same equation. Finding the roots is not easy. Try a numerical scheme.

6. $p(x) = ax^2 + bx + c$ leads to $c = 0$ $\quad a = \sqrt{2}/2 - 1 \quad b = 1 - a = 2 - \sqrt{2}/2$

7. $p(x) = 2f(0) - 4f(1/2) + 2f(1)$. Use this to find an approximation for area under the function
$$\text{Area} = \int_0^1 p(x) = \frac{1}{6}f(0) + \frac{2}{3}f(1/2) + \frac{1}{6}f(1)$$
which is Simpson's rule (look this up if you have not covered it yet).

8. Symmetry tells us that $p(1/2) = 1/2$ and there are three roots, one of which is repeated; $p(1) = 1$ and $p(1/2) = 1/2$ to give $p(x) = x(4x - 3)^2$.

9. $2x^4 + 15x^3 + 32x^2 + 15x + 2 = (x^2 + 4x + 1)(2x^2 + 7x + 2)$

10. $x = 6, \quad y = 30$, or $x = 1, \quad y = 10$, or $x = 30, \quad y = 6$, or $x = 4, \quad y = -20$, or $x = -20, \quad y = 4$

11. Start by finding p and q such that we can write the general cubic equation $ax^3 + bx^2cx + d = 0$ in the form $y^3 + py + q = 0$. Then find a new a and b such that $y^3 + py + q = y^3 + a^3 + b^3 - 3aby$

Exercise 10.1 Solutions

1. Use $z_1 = x + iy$, $z_2 = u + iv$ and then show that $z_1 + z_2 = z_2 + z_1$

2. Show $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$ also perhaps show that $k(z_1 + z_2) = kz_1 + kz_2$ where k is constant

3. $i^5 = i$, $i^7 = -i$, $i^9 = i$, $,i^{-4} = 1$, $,i^{-5} = -i$, $,i^{4n} = 1$, $,i^{4n+1} = i$

4. i) $\bar{z} = a - ib$
   ii) $z + \bar{z} = 2a$
   iii) $z - \bar{z} = 2ib$
   iv) $z\bar{z} = a^2 + b^2$

5. If $z = \dfrac{\sqrt{2}}{2}(1 + i)$ then $z^2 = i$

6. If $z_1 = 1 + i$, $z_2 = 2 + 3i$, and $z_3 = 4 - 2i$ then
   i) $z_1 + z_2 = 3 + 4i$
   ii) $z_3 - z_1 = 3 - 3i$
   iii) $(3 + 4i)z_1 = -1 + 7i$
   iv) $z_2z_3 = 14 + 8i$
   v) $\dfrac{5 - i}{z_1} = 2 - 3i$
   vi) $z_1\bar{z}_1 = 2$
   vii) $\dfrac{z_2}{\bar{z}_3} = \dfrac{7}{10} + \dfrac{2}{5}i = 0.7 + 0.4i$
   viii) $\dfrac{\bar{z}_1z_2}{z_3} = 0.9 + 0.7i$
   ix) $\dfrac{(z_1 + z_3)z_2}{z_3 - z_2} = \dfrac{13}{29}(-3 + 7i)$
   x) $\dfrac{(z_1 + z_2)^2}{(42z_1 - 11z_2 + z_3)} = i$

7. i) $(1+i)^3 = -2+2i$

ii) $(1-i)^3 = -2-2i$

8. i) $\frac{2z}{1+i} - \frac{2z}{i} = \frac{5}{2+i} \quad \Rightarrow z = \frac{1-3i}{2}$

ii) $\frac{z}{1+2i} - \frac{2z}{i-1} = 3 \quad \Rightarrow z = 2-i$

9. $\frac{1}{z} + \frac{1}{z+6i} = \frac{4}{13} \quad \Rightarrow z = \frac{9}{2} - 3i$ or $z = 2-3i$ by direct use of quadratic formula, or use the substitution $w = z + 3i$.

10. i) $\sqrt{3+4i} = \pm(2+i)$

ii) $\sqrt{i} = \pm\frac{\sqrt{2}}{2}(1+i)$

11. $u = \frac{z+\bar{z}}{2}$, $v = \frac{z-\bar{z}}{2i} = \frac{1}{2}(\bar{z}-z)$, $u^2 + v^2 = z\bar{z}$

12. $z^3 + \bar{z} = 0 \quad \Rightarrow z = 0, \pm\frac{1}{\sqrt{2}}(1+i), \pm\frac{1}{\sqrt{2}}(1-i)$

13. i) $\arg(z) = \operatorname{Arg}(z) + 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots.$ is true since the tangent function is periodic with period 2π

ii) $\operatorname{Arg}(\bar{z}) = -\operatorname{Arg}(z)$ true by definition of the conjugate use $z = x + iy$

iii) $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$ true use $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

iv) $\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg}(z_1 z_2)$ true use $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then consider the result to be true and take the tangent of both sides of the equation.

Exercise 10.2 Solutions

1. i) $|4+3i|=5$

   ii) $|1+i\sqrt{3}|=2$

   iii) $|1-i\sqrt{3}|=2$

2. i) $z_1z_2=-7+22i$ and $|z_1z_2|=23.09$ and note that $=|z_1||z_2|$

   ii) $\dfrac{z_1}{z_2}=\dfrac{1}{41}(23+2i)$ and $\left|\dfrac{z_1}{z_2}\right|=0.56$ and note that $=\dfrac{|z_1|}{|z_2|}$

3. Theoretical work proving the observations in Question 2.

4. i) $|5z_1|=10$

   ii) $|-z_2|=3$

   iii) $|-3\bar{z}_1|=6$

   iv) $|z_1z_2|=6$

   v) $\left|\dfrac{i\bar{z}_1}{2z_2}\right|=\dfrac{1}{3}$

5. Based on a problem from electrical engineering. Have a go.

6. $z^3+\bar{z}=0 \Rightarrow z=0, \pm\dfrac{1}{\sqrt{2}}(1+i), \pm\dfrac{1}{\sqrt{2}}(1-i)$. Hence $z=0$ or $|z|=1$.

7. Use $z=x+iy$.

8. $f(z)$ is linear if for two numbers z_1 and z_2 then if $f(z_1)=f_1$ and $f(z_2)=f_2$ we have $f(z_1+z_2)=f_1+f_2$

Exercise 10.3 Solutions

1. i) $|z-1+i|=3$ is a circle centre $(1,-i)$ radius 3. Use triangles to visualise $|z-1+i|$ geometrically

ii) $|z-z_1|=r$ z_1 fixed and r real is a circle centre $z_1=(x_1,y_1)$ radius r

iii) $|z-1|^2+|z-2|^2=4$. Complete the square. Circle radius $\frac{\sqrt{7}}{2}$, centre $\left(\frac{3}{2},0\right)$

iv) $\left|\frac{z-i}{z+i}\right|=1$ leads to the x axis, $y=0$

v) $2\text{Re}(z)+\text{Im}(z)=1$ solutions lie on the line $y=-2x+1$

Exercise 11 .1 Solutions

1. i) $x^2 + 25 = 0 \ \Rightarrow x = \pm 5i$

   ii) $4x^2 + 4x + 5 = 0 \ \Rightarrow x = -\frac{1}{2} \pm i$

   iii) $x^2 + 2x + 2 = 0 \ \Rightarrow x = -1 \pm i$

   iv) $x^2 + 4x + 13 = 0 \ \Rightarrow x = -2 \pm 3i$

   v) $2x^2 + x + 5 = 0 \ \Rightarrow x = \frac{-1 \pm i\sqrt{39}}{4}$

   vi) $x^2 + 2x + 5 = 0 \ \Rightarrow x = -1 \pm 2i$

   vii) $x^4 - 3x^2 - 4 = 0 \ \Rightarrow x = \pm 2, \ \pm i$

   viii) $x^3 + x - 2 = 0 \ \Rightarrow x = 1, \ -\frac{1}{2} \pm i\frac{\sqrt{7}}{2}$

2. i) $(x - 6i)(x + 6i) = x^2 + 36 = 0$

   ii) $x^2 - 16x + 100 = 0$

   iii) $x^4 + 1 = 0$

3. Square z and plug it into the equation.

4. $x^4 = 81 \ \Rightarrow x = \pm 3, \ 3i$

5. i) $z^3 + z^2 + z + 1 = 0 \ \Rightarrow z = -1, \pm i$

   ii) $(z + 3)^3 = 8 \ \Rightarrow z = -1, -4 \pm i\sqrt{3}$

6. $z^2 + z + 1 = 0 \ \Rightarrow z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ then square them.

7. $x^4 + 1 = 0 \ \Rightarrow x = \pm \frac{\sqrt{2}}{2}(1 + i), \ \pm \frac{\sqrt{2}}{2}(1 - i)$

Exercise 12.1 Solutions

Note : checking that you have the right answer is fairly easy and always worthwhile.

1. i) $$\frac{1}{1-t^2}=\frac{1}{2(1-t)}+\frac{1}{2(1+t)}$$

ii) $$\frac{2}{x^2-1}=\frac{1}{x-1}-\frac{1}{x+1}$$

iii) $$\frac{4}{2x-x^2+3}=\frac{1}{x+1}+\frac{1}{3-x}$$

iv) $$\frac{3x+1}{x^2+x-2}=\frac{4}{3(x-1)}+\frac{5}{3(x+2)}$$

v) $$\frac{1}{x^2-5x+6}=\frac{1}{x-3}-\frac{1}{x-2}$$

vi) $$\frac{1}{x^2+x}=\frac{1}{x}-\frac{1}{x+1}$$

vii) $$\frac{3x+2}{x^2-2x}=\frac{4}{x-2}-\frac{1}{x}$$

viii) $$\frac{x^3}{x^2-3x+2}=x+3+\frac{8}{x-2}-\frac{1}{x-1}$$

ix) $$\frac{5(x+1)}{25-x^2}=\frac{3}{5-x}-\frac{2}{5+x}$$

x) $$\frac{x^3}{x^2+x-2}=x-1+\frac{1}{3(x-1)}+\frac{8}{3(x+2)}$$

xi) $$\frac{x^2+2x+1}{(x-1)(x+2)(x+3)}=\frac{1}{3(x-1)}-\frac{1}{3(x+2)}+\frac{1}{x+3}$$

2. $\dfrac{1}{x(x+1)(x+3)}=\dfrac{1}{3x}-\dfrac{1}{2(x+1)}+\dfrac{1}{6(x+3)}$ and so the series sums to

$$\frac{1}{1.2.4}+\frac{1}{2.3.5}+\frac{1}{3.4.6}+....\frac{1}{n(n+1)(n+3)}=\frac{7}{36}+\frac{1}{6(n+2)}+\frac{1}{6(n+3)}-\frac{1}{3(n+1)}$$

and so as $n\to\infty$ the sum tends to $\dfrac{7}{36}$

3. i) $$\frac{1}{x^3-2x^2+x}=\frac{1}{x}-\frac{1}{x-1}+\frac{1}{(x-1)^2}$$

ii) $$\frac{2x}{x^3-3x+2}=\frac{4}{9(x-1)}+\frac{2}{3(x-1)^2}-\frac{4}{9(x+2)}$$

iii) $$\frac{4x}{x^3+x^2+4x+4}=\frac{4(x+4)}{5(x^2+4)}-\frac{4}{5(x+1)}$$

iv) $$\frac{x^2+4x+5}{x^2+4x+3}=1+\frac{1}{x+1}-\frac{1}{x+3}$$

v) $$\frac{x^2}{x^3+3x^2+3x+1}=\frac{1}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{(x+1)^3}$$

vi) $$\frac{16x}{x^4-16}=\frac{1}{x-2}+\frac{1}{x+2}-\frac{2x}{x^2+4}$$

vii) $$\frac{x^2+x+1}{x^2+2x+1}=1-\frac{1}{x+1}+\frac{1}{(x+1)^2}$$

viii) $$\frac{10-11x}{x^3-4x^2+x-4}=\frac{2x-3}{x^2+1}-\frac{2}{x-4}$$

ix) $$\frac{3x^3+x}{x^4-81}=\frac{7}{9(x-3)}+\frac{7}{9(x+3)}+\frac{13x}{9(x^2+9)}$$

x) $$\frac{2x^2-11x+5}{x^3-x^2-11x+15}=\frac{3x}{x^2+2x-5}-\frac{1}{x-3}$$

xi) $$\frac{2y+1}{y^3-3y-2}=\frac{5}{9(y-2)}-\frac{5}{9(y+1)}+\frac{1}{3(y+1)^2}$$

xii) $$\frac{5x+3}{2x^3+5x^2+4x+1}=\frac{2}{(x+1)^2}-\frac{1}{x+1}+\frac{2}{2x+1}$$

xiii) $$\frac{5x^3+2x^2+5x}{x^4-1}=\frac{3}{x-1}+\frac{2}{x+1}+\frac{1}{x^2+1}$$

4. i) $$\frac{p^2}{(p^2+1)(p^2+2)}=\frac{2}{p^2+2}-\frac{1}{p^2+1}$$

ii) $$\frac{1}{x(x-1)^3}=\frac{1}{x-1}-\frac{1}{x}-\frac{1}{(x-1)^2}+\frac{1}{(x-1)^3}$$

iii) $$\frac{x}{(x-1)(x+2)(2x+3)^2}=\frac{1}{75(x-1)}+\frac{2}{3(x+2)}-\frac{29}{25(2x+3)}+\frac{3}{5(2x+3)^2}$$

Exercise 13.1 Solutions

1. i) $n > 1000$

   ii) $n > 31$

2. i) $\left\{\frac{1}{n}\right\} \to 0$

   ii $\left\{1 + \frac{1}{n}\right\} \to 1$

   iii) $\left\{\frac{7n^2 - 1}{n^2}\right\} \to 7$

   iv) $\left\{\frac{n+1}{n}\right\} \to 1$

   v) $\left\{\frac{1}{n}\cos\left(\frac{n\pi}{\sqrt{2}}\right)\right\} \to 0$

   vi) limit of $\{(-1)^n\}$ does not exist

   vii) $\left\{1 + \frac{1}{2^n}\right\} \to 1$

   viii) $\left\{\frac{n+7}{4n+2}\right\} \to \frac{1}{4}$

   ix) $\left\{(-1)^{n^2}\right\} = -1, +1, -1, +1, \ldots.$ and so does not converge

3. Own sequence, for example $\{S_n\} = \left\{\frac{4n^2 + 63}{2 + n^2}\right\}$ then $\lim_{n\to\infty} S_n = 4$.

4. $\{(-1)^n\}$ is bounded but the sequence does not converge. We require bounded plus monotonically increasing.

5. $\{S_n\} = \left\{1 - \frac{1}{n^2}\right\} = \left\{0, \frac{3}{4}, \frac{8}{9}, \frac{15}{16}, \ldots\right\}$ the terms increase and the sequence converges to 1.

Exercise 13.2 Solutions

1. Essentially looking at the comparison test for series with positive terms

i) $\frac{1}{n(n+1)} < \frac{1}{n^2}$ and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ also converges

ii) $1 + \frac{1}{2} + \frac{1}{3.2} + \frac{1}{4.3.2} + \ldots. < 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots.$ which is a geometric series that converges hence $\sum_{n=1}^{\infty} \frac{1}{n!}$ also converges

iii) $\frac{2n+1}{2n^2-3} > \frac{2n+1}{2n^2} > \frac{2n}{2n^2} = \frac{1}{n}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges hence $\sum_{n=1}^{\infty} \frac{2n+1}{2n^2-3}$ diverges.

2. Ratio test. For a series with positive terms if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = p$, then series converges if $p < 1$ and diverges if $p > 1$.

i) $\frac{u_{n+1}}{u_n} = \frac{n}{n+1} \times \frac{1}{x} \quad \Rightarrow \lim_{n\to\infty} = \frac{1}{x}$

$\sum_{n=1}^{\infty} \frac{1}{nx^n}$ converges for $x > 1$, diverges for $x < 1$

ii) $\frac{u_{n+1}}{u_n} = \frac{n+1}{10} \quad \Rightarrow$ as $n \to \infty$, $\frac{u_{n+1}}{u_n} \to \infty$ and hence $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ diverges

iii) $\frac{u_{n+1}}{u_n} = \frac{x}{n} \quad \Rightarrow \lim_{n\to\infty} = 0$ so series always converges

3. i) $(1+2x)^5 = 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$

Exercise 14.1 Solutions

1. Assuming that

$$e = 1 + 1 + \frac{1}{2!} + \quad \ldots. + \frac{1}{n!} + \ldots.$$

$$= 1 + 1 + \frac{1}{2!} + \ldots. \quad \ldots. + \frac{1}{n!} + \frac{1}{(n+1)!}\left\{1 + \frac{1}{n+2} + \frac{1}{(n+3)(n+2)} + \ldots. \quad \right\}$$

then considering the terms inside the bracket $\frac{1}{n+2} < \frac{1}{n+1}$, and $\frac{1}{(n+3)(n+2)} < \frac{1}{(n+1)^2}$ etc. so that the expression inside the bracket is less than $\left\{1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \ldots. \quad \right\}$ which is a G.P. which we can sum and replace the bracket by $\frac{n+1}{n}$. Hence, after truncating the infinite series we can bound e as follows

$$1 + 1 + \frac{1}{2!} + \ldots. \quad \ldots. + \frac{1}{n!} < e < 1 + 1 + \frac{1}{2!} + \ldots. \quad \ldots. + \frac{1}{n!} + \frac{1}{(n+1)!}\frac{n+1}{n}$$

Now assume that e is rational and the result follows.

2. We know that $\pi > e$. Let $\pi = \delta e$ for some positive δ and then take logs.

3. $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$. Think about the derivative of exp(x).

4. Try and separate the general term into two factors that will lead to something related to the series for e. The answer is 2e.

5. As above but now the answer is 1.

iii) $0.99 = 1 - 0.01 \quad \Rightarrow (0.99)^4 \approx 0.96$

iv) $(9 + x)^{1/2} = 3 + \frac{x}{6} + \frac{x^2}{216} + \frac{x^3}{3888} + \ldots.$

v) $\left(\sqrt{3} + 1\right)^3 \approx 20.39$

Exercise 14.2 Solutions

1. Use $\cosh(x)+\sinh(x)=e^x$ etc. plus basic identities

2. Use basic definition $\sinh(x+y)=\dfrac{e^{(x+y)}-e^{-(x+y)}}{2}$ etc.

3. You can use the results from Question 2.

4. The aim here is to use the results from Question 2.

5. Known as Osborne's rule : Use the same expression as for circular functions but change the sign in front of the product of two sines. For example

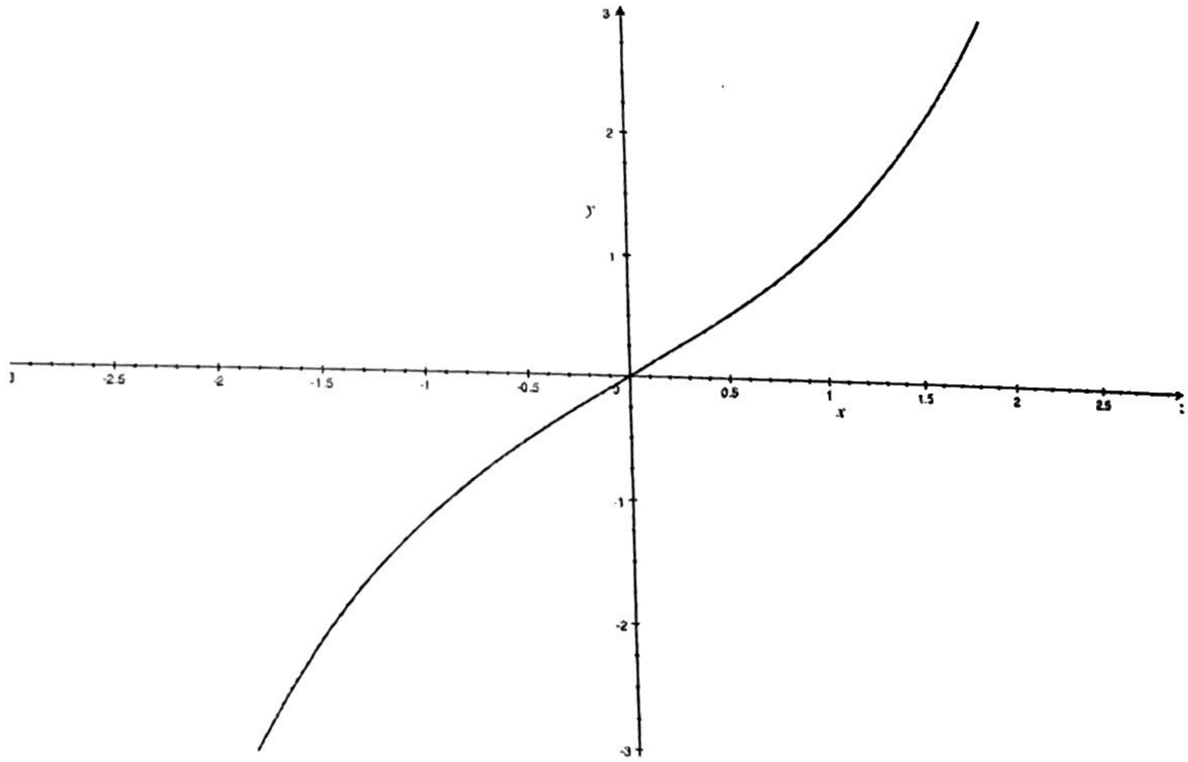
$$\sin(3A)=3\sin(A)-4\sin^3(A)$$

becomes

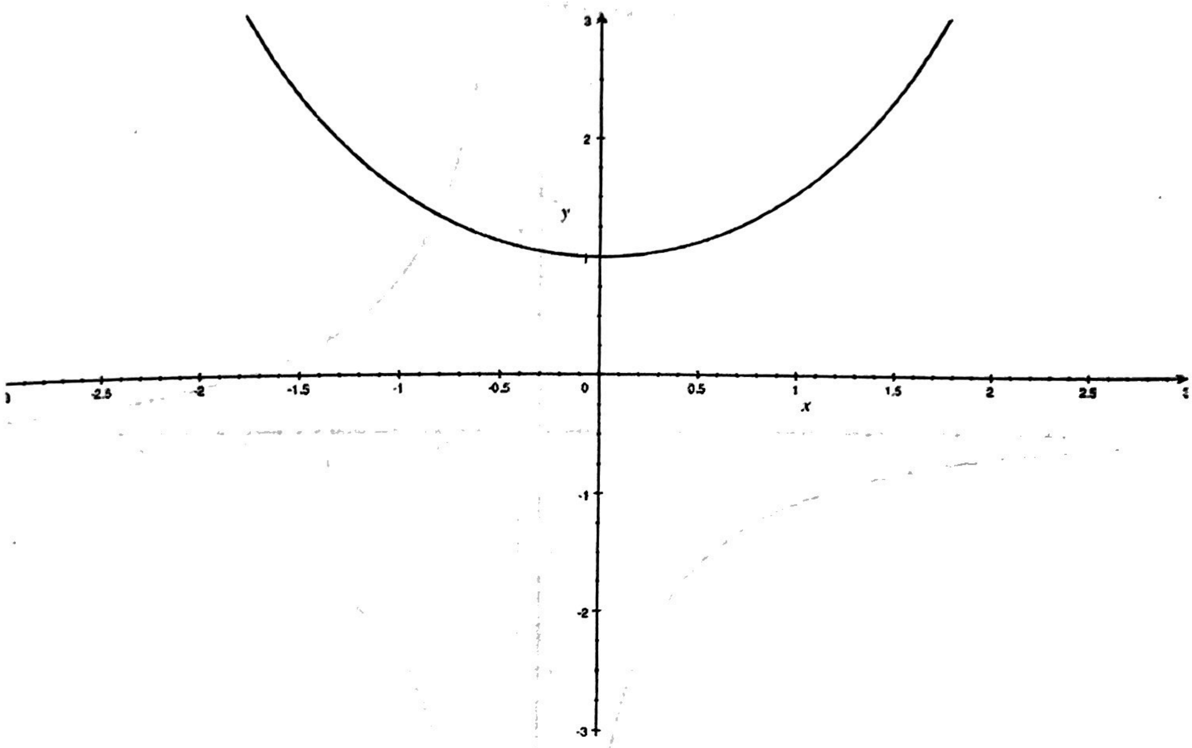
$$\sinh(3A)=3\sinh(A)+4\sinh^3(A)$$

6. First let is plot the hyperbolic functions

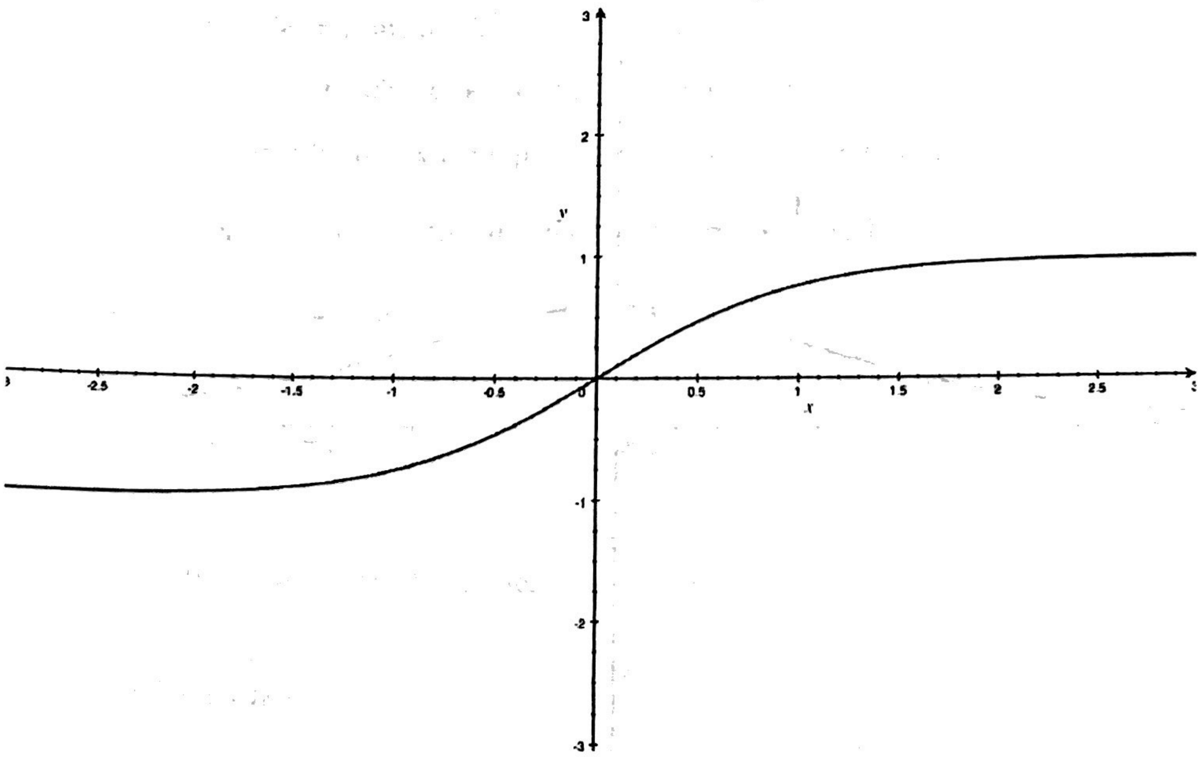
$y=\sinh(x)$



$y = \cosh(x)$

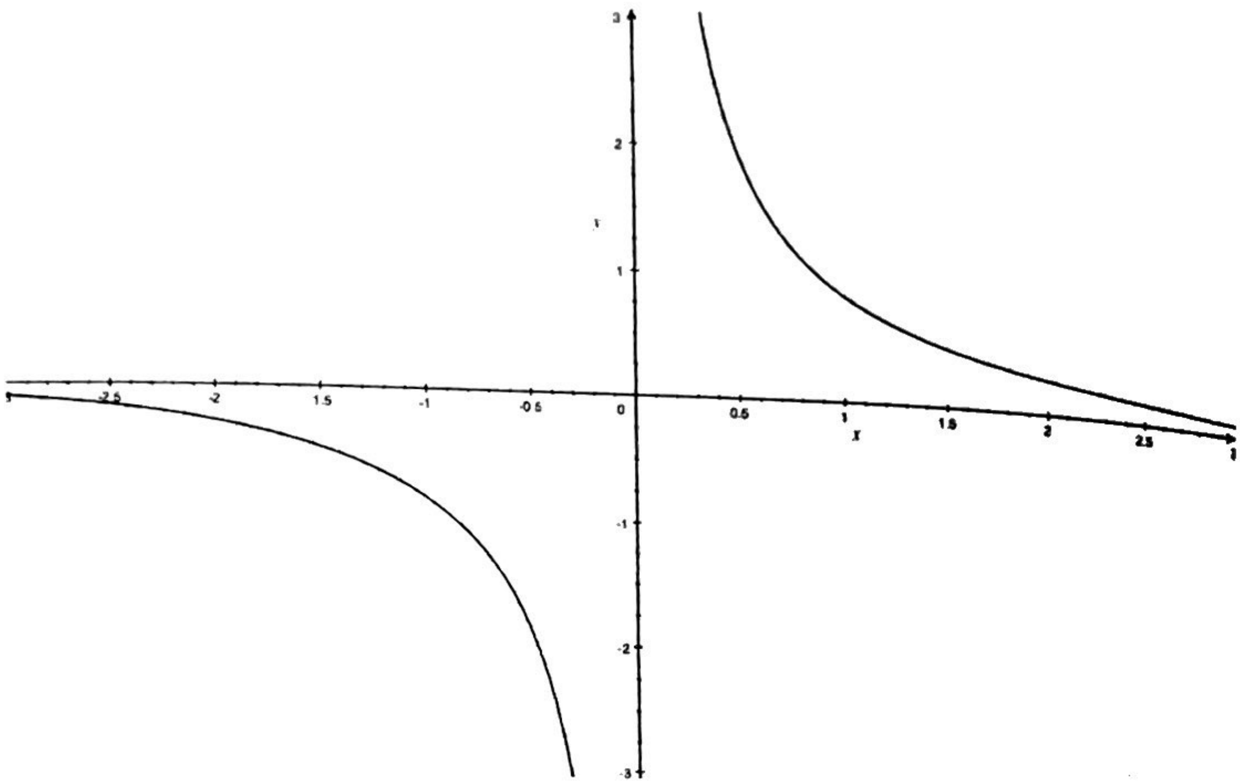


$y = \tanh(x)$

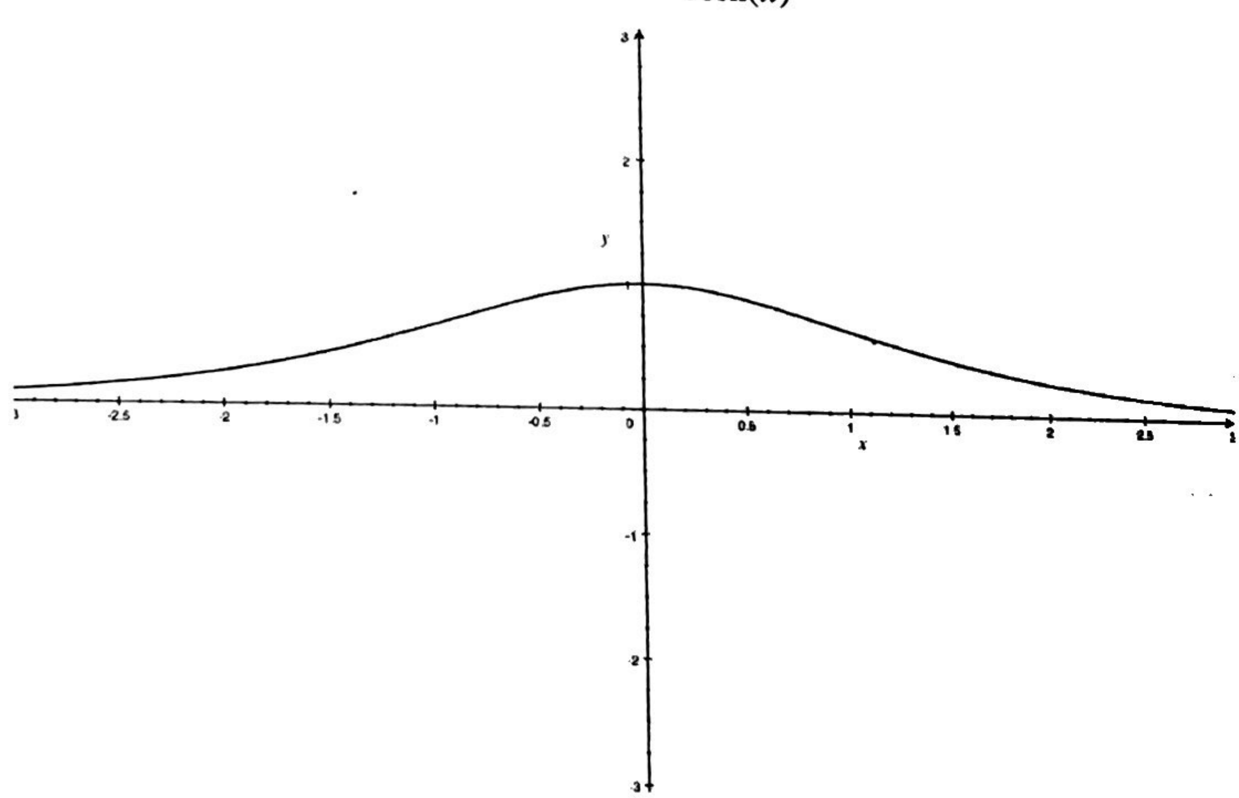


Sketch of the functions cosech(x), sech(x) and coth(x)

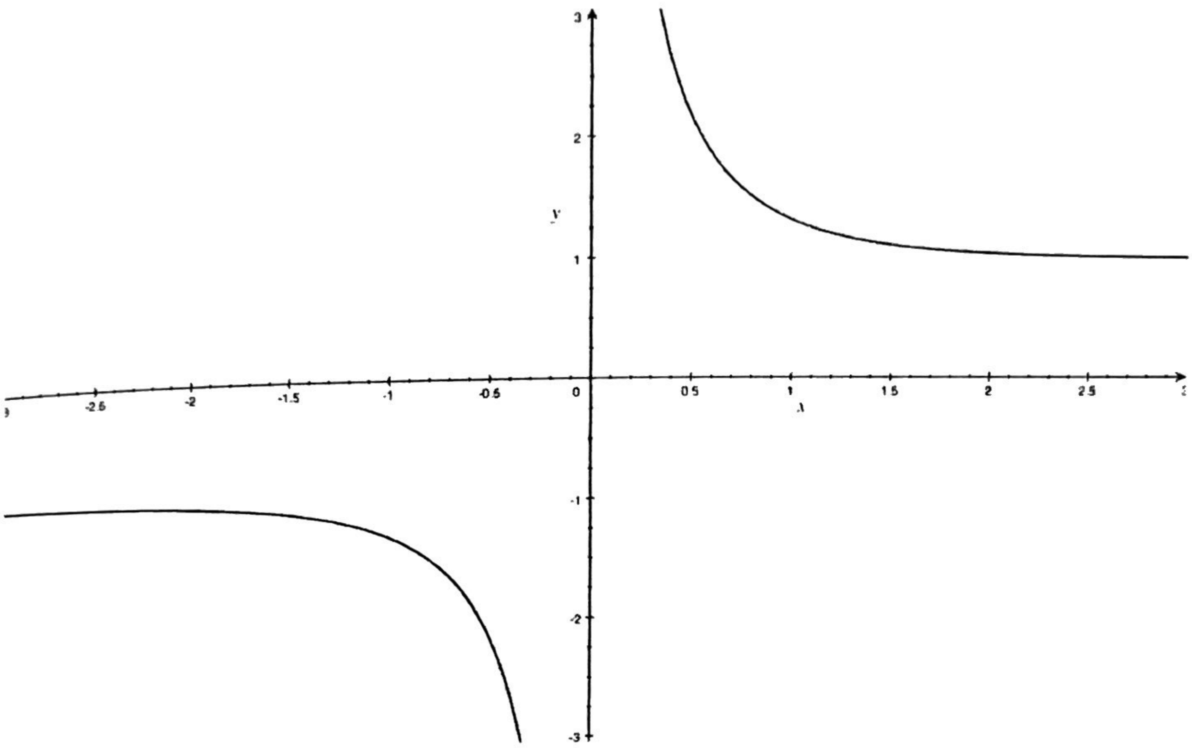
$$y = \text{cosech}(x) = \frac{1}{\sinh(x)}$$



$$y = \text{sech}(x) = \frac{1}{\cosh(x)}$$



$$y = \coth(x) = \frac{1}{\tanh(x)}$$



7. i) Use definitions in terms of e^x to show that

$$4\tanh^2(x) + 3 = 8\tanh(x) \Rightarrow e^{2x} = 3$$

other solutions are not possible since $e^x > 0$. We have not covered logarithms yet here but $\Rightarrow e^{2x} = 3 \quad \Rightarrow x = \frac{1}{2}\log 3$

ii) $\sinh(x) = 1 + e^{-x} \quad \Rightarrow e^x = 3 \quad \Rightarrow x = \log 3$

iii) $5\cosh(x) - 3\sinh(x) = 5 \quad \Rightarrow x = 0 \; or \; e^x = 4 \quad \Rightarrow x = 2\log 2$

iv) $5\sinh(x) + 3\sinh(x) = -3 \quad \Rightarrow e^x = \frac{1}{4} \quad \Rightarrow x = -2\log 2$

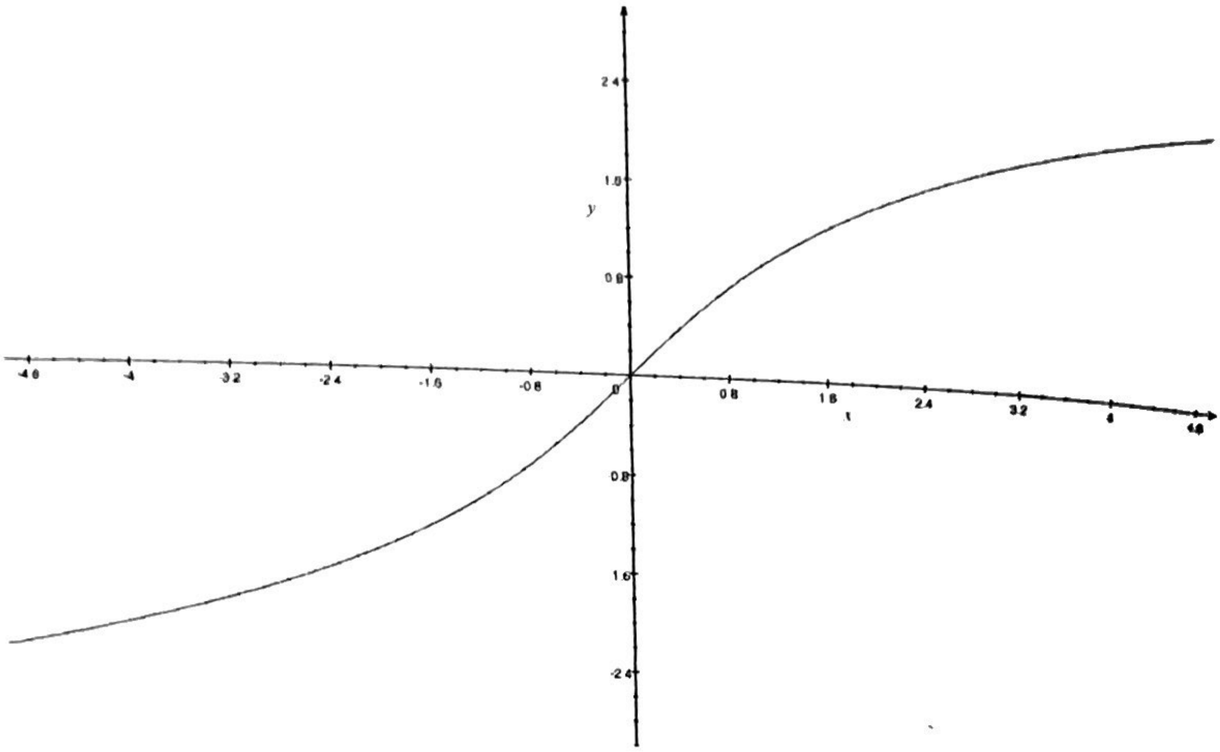
8. Use definition in terms of exponential function.

9. Consider $\cosh(x) - \sinh(x)$.

Exercise 14.3 Solutions

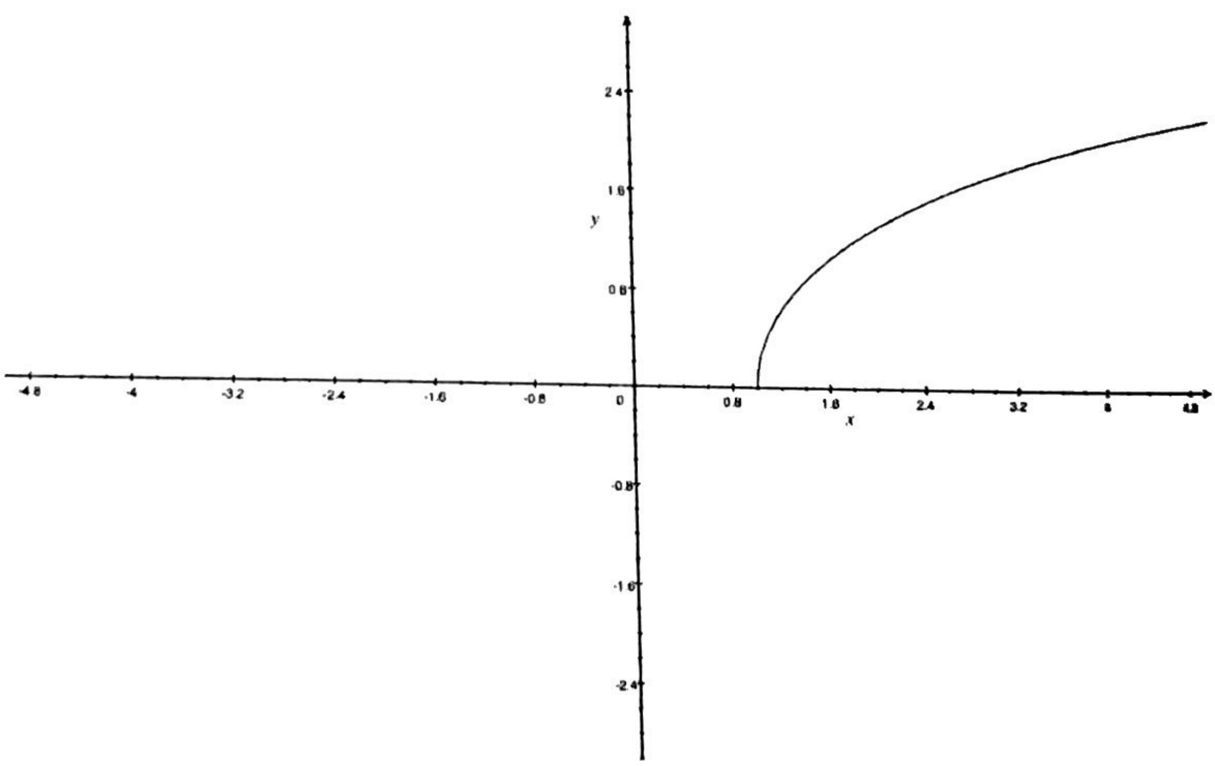
1. Plot of the inverse hyperbolic functions by reflecting in the $y = x$ line.

   i) $y = \sinh^{-1}(x)$

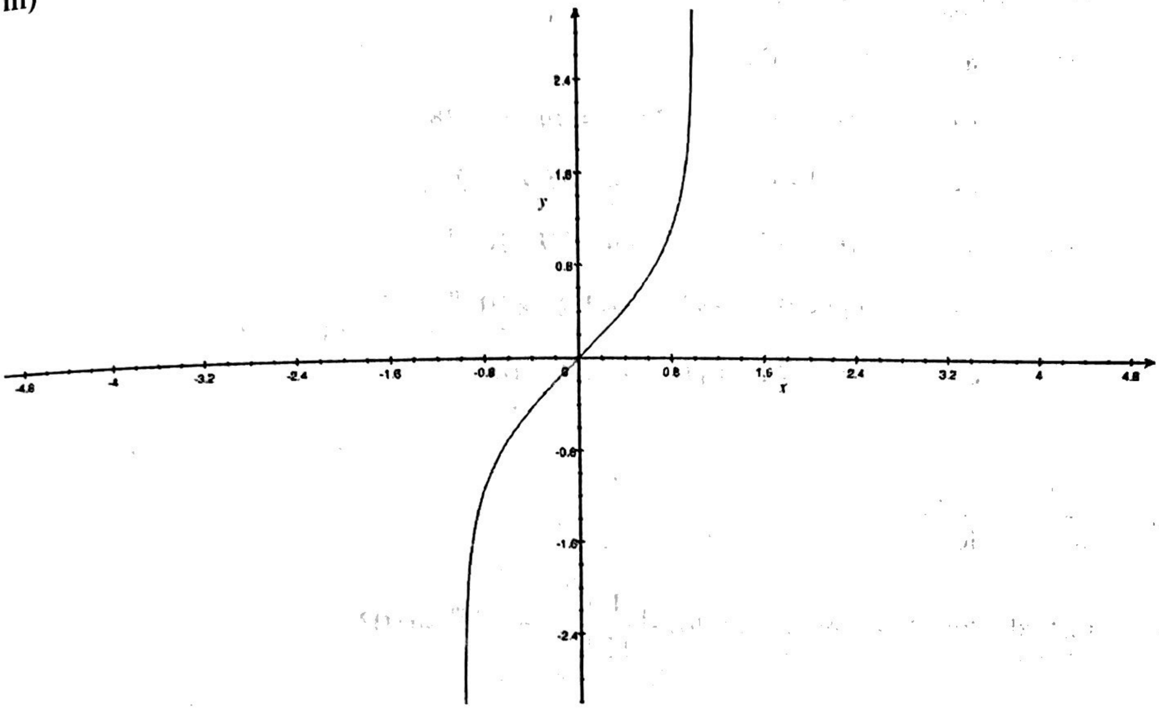


   ii) $y = \text{Cosh}^{-1}(x)$ For inverse coshine we need to take principal value to define a function.

$y = \text{Cosh}^{-1}(x)$



iii) $y = \tanh^{-1}(x)$



Exercise 15.1 Solutions

1. $pH = -\log_{10}[H^+]$, so use logs to the base 10

   i) a) $[H^+] = 0.000275 \Rightarrow pH \approx 3.56$

   b) $[H^+] = 0.0000525 \Rightarrow pH \approx 4.28$

   c) $[H^+] = 6.75 \times 10^{-9} \Rightarrow pH \approx 8.17$

   ii) a) $pH = 9.23 \Rightarrow [H^+] \approx 5.89 \times 10^{-10}$

   b) $pH = 9.84 \Rightarrow [H^+] \approx 1.45 \times 10^{-10}$

   c) $pH = 7.5 \Rightarrow [H^+] \approx 3.16 \times 10^{-8}$

2. $\frac{1}{\log_2 n} + \frac{1}{\log_3 n} + ... + \frac{1}{\log_{10} n} = \frac{1}{\log_{10!} n}$

3. Just take logs but now natural logs $\left(\frac{1}{12}\right)^{1.405} = e^{-3.49} \approx 0.03.$

4. i) $\ln\left(\sqrt{e}\right) = \frac{1}{2}$

   ii) $\ln\left(e^3\right) = 3$

5. i) $4 = 9e^{2x} \Rightarrow x \approx -0.41$

   ii) $\ln(x^2 + 2x) = 2.07944 \Rightarrow x = -4, 2$

   iii) $7 = 91e^{-2.1t} \Rightarrow t \approx 1.22$

   iv) $960 = 423 \times (2.28)^{0.63} \times 14^x \Rightarrow x \approx 0.11$

   v) $3^{2x} = 5^{x+1} \Rightarrow x \approx 2.74$

   vi) $5^{2x} - 5^{x+1} + 4 = 0 \Rightarrow x = 0$ or $x \approx 0.86$

   vii) $3\cosh(x) - 5\sinh(x) = 0 \Rightarrow x = \ln 2 \approx 0.69$

6. i) $x = 1$ then $x^{1/x} = 1$

   ii) $x = 10$ then $x^{1/x} = 1.25$

   iii) $x = 100$ then $x^{1/x} \approx 1.047$

   iv) $x = 500$ then $x^{1/x} \approx 1.0125$

   v) $x = 1000$ then $x^{1/x} \approx 1.0069$ to see that $x^{1/x} \to 1$ as $x \to \infty$, but slowly

7. Write as $-\ln\left(1-\frac{1}{x^2}\right)$ to give $\ln\left(\frac{x^2}{x^2-1}\right)=\frac{1}{x^2}+\frac{1}{2x^4}+\frac{1}{3x^6}+\ldots$

8. i) $\ln(1+3x)=3x-\frac{9x^2}{2}+\frac{27x^3}{3}-\frac{81x^4}{4}+\ldots \quad -\tfrac{1}{3}<x<\tfrac{1}{3}$

ii) $\ln\left(1-\tfrac{x}{3}\right)=-\left\{\frac{x}{3}+\frac{x^2}{18}+\frac{x^3}{81}+\frac{x^4}{324}+\ldots\right\} \quad -3<x<3$

iii) $\ln\left(1+\tfrac{x}{3}\right)=\frac{x}{3}-\frac{x^2}{18}+\frac{x^3}{81}-\frac{x^4}{324}+\ldots \quad -3<x<3$

iv) $\ln(3+x)=\ln 3+\frac{x}{3}-\frac{x^2}{18}+\frac{x^3}{81}-\frac{x^4}{324}+\ldots \quad -3<x<3$

v) $\ln(4+x)^3=3\left\{\log 4+\frac{x}{4}-\frac{x^2}{32}+\frac{x^3}{192}-\frac{x^4}{1024}+\ldots\right\} \quad -4<x<4$

vi) $\ln\sqrt{(1-x-2x^2)}=-\frac{1}{2}\left\{x+\frac{5x^2}{2}+\frac{7x^3}{3}+\frac{17x^4}{4}+\ldots\right\} \quad -\tfrac{1}{2}<x<1$

vii) $\ln\left(\frac{1+x}{1-x}\right)=2\left\{x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\ldots\right\} \quad -1<x<1$

9. Use logs to the base 10 (or base 2) $x\approx 1262611$

10. $\sum_{k=1}^{n}\ln k=\ln(n!)$

11. $\left(\frac{1-x}{x}\right)\ln(1-x)+1$

Exercise 15.2 Solutions

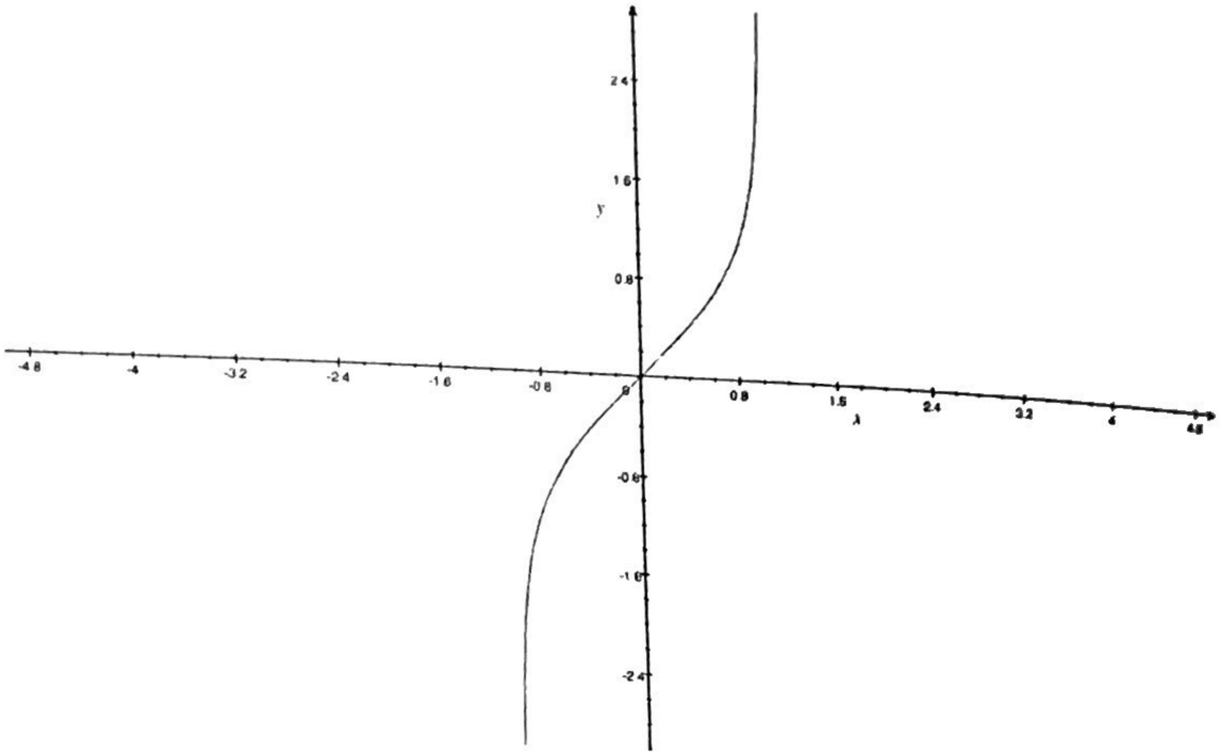
1. From the graphs of the functions

   i) $\text{Cosh}^{-1}(1) = 0$

   ii) $\sinh^{-1}(0) = 0$

2. Sketch of the graph of $\tanh^{-1}(x)$

$y = \tanh^{-1}(x)$



3. To show that $\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ $|x| < 1$ use the definition of $\tanh(y)$ in terms of the exponential function.

4. $\text{sech}(y) = x$ with $y \geq 0$ and to show that $\text{Sech}^{-1}(x) = \ln\left(\frac{1+\sqrt{(1-x^2}}{x}\right)$ $0 < x \leq 1$ use the definition of $\cosh(y)$ in terms of the exponential function.

5. i) Use $\cosh^2 - \sinh^2 = 1$ or formula to show $\sinh^{-1}\left(3/4\right) = \ln 2 \approx 0.693$

   ii) $\sinh^{-1}\left(4/3\right) = \ln 3 \approx 1.099$

6. $y = \text{Cosh}^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x + \sqrt{(x^2 - a^2}}{a}\right\}$. Hence $\text{Cosh}^{-1}\left(\frac{13}{5}\right) = \ln 5 \approx 1.609$.

7. $y = \tanh^{-1}\left(\frac{x}{a}\right) = \frac{1}{2}\ln\left(\frac{a+x}{a-x}\right)$. Hence $\text{Cosh}^{-1}\left(\frac{3}{5}\right) = \ln 2 \approx 0.693$.

8. i) $\text{Cosh}^{-1}\left(\frac{5}{4}\right) = \ln 2 \approx 0.693$

   ii) $\tanh^{-1}\left(\frac{2}{7}\right) = \frac{1}{2}\ln\left(\frac{9}{5}\right) \approx 0.294$

   iii) $\sinh^{-1}(0.6) \approx \ln 1.766 \approx 0.569$

   iv) $\text{Cosh}^{-1}(3) = \ln\left(3 + \sqrt{8}\right) \approx 1.763$

   v) $\tanh^{-1}(0.7) = \frac{1}{2}\ln\left(\frac{1.7}{0.3}\right) \approx 0.867$.

9. Also see Exercise 14.2 Question 7 part (i).

To recap, in terms of the logarithmic function.

$y = \sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$

$y = \cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right) \quad x \geq 1$

$y = \tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad |x| < 1$

$y = \text{cosech}^{-1}(x) = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$

$y = \text{sech}^{-1}(x) = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \quad 0 < x \leq 1$

$y = \coth^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) \quad |x| > 1$

Exercise 16.1 Solutions

1. Let $y = \dfrac{1}{\sqrt{x}}$ $x = 9, \delta x = 0.01$. Square both sides and ignore second order terms.

   Then $\dfrac{1}{\sqrt{9.01}} \approx 0.3331483$ accurate to 7 decimal places.

2. Seems obvious but use the sum of a geometric progression.

3. Use $f'(a) \approx \dfrac{f(a+\delta x) - f(a)}{\delta x}$

   i) $f = e^{x^{1/3}}$ $\Rightarrow f'(1.2) \approx 0.854288$

   ii) $f = \ln\left\{ x \sin\left(\dfrac{1}{x^2}\right)\right\}$ $\Rightarrow f'(3) \approx -0.33059$

Exercise 16.2 Solutions

1. From first principles. Seems like hard work but it lets the student know that if in doubt they can always resort to first principles, but it also shows that it is worth remembering the rules for the basic functions - it saves a lot of time.

2. Write the hyperbolic sine and hyperbolic cosine functions in terms of exponential functions.

3. Plot of the function. For continuity we need

   Limit from left = limit from right = value of function at the point

   The function at $x = 1$ is then continuous but not differentiable.

Exercise 16.3 Solutions

1. Not so straightforward as you might imagine. Strictly we should consider the case when $\frac{du}{dx} \neq 0$ and $\frac{du}{dx} = 0$.

2. Straightforward from first principles.

3. So really we only need the product rule plus the function of a function rule.

4. i) $\frac{d}{dx}\{\sin(ax+b)\} = a\cos(ax+b)$

   ii) $\frac{d}{dx}\left\{e^{\sin(x^2)}\right\} = 2x\cos(x^2)e^{\sin(x^2)}$

5. The function is continuous at $x = 0$ but not differentiable.

6. To differentiate these functions use the quotient rule.

Exercise 16.4 Solutions

1. i) $y = \cos^3(x) \quad \Rightarrow y' = -3\cos^2(x)\cdot\sin(x)$

   ii) $y = (4x-5)^6 \quad \Rightarrow y' = 24(4x-5)^5$

2. i) $y = (3x^2+1)^2 \quad \Rightarrow y' = 12x(3x^2+1)$

   ii) $y = \dfrac{1}{2x+7} \quad \Rightarrow y' = \dfrac{-2}{(2x+7)^2}$

   iii) $y = \sqrt{(2-3x)} \quad \Rightarrow y' = \dfrac{-3}{2\sqrt{(2-3x)}}$

   iv) $y = \dfrac{3x-1}{2x-15} \quad \Rightarrow y' = \dfrac{-28}{(2x-15)^2}$

   v) $y = \left(\dfrac{x-1}{x}\right)^6 \quad \Rightarrow y' = -\dfrac{6(x-1)^5}{x^7}$

   vi) $y = \sin(2x) \quad \Rightarrow y' = 2\cos(2x)$

   vii) $y = \cos^3(4x) \quad \Rightarrow y' = -12\sin(4x)\cos^2(4x)$

   viii) $y = \cos(\sin(x)) \quad \Rightarrow y' = -\cos(x)\sin(\sin(x))$

   ix) $y = \dfrac{2x+3}{\sin(x)} \quad \Rightarrow y' = \dfrac{2\sin(x)-(2x+3)\cos(x)}{\sin^2(x)}$

   x) $y = \dfrac{\sin^2(2x)}{x} \quad \Rightarrow y' = \dfrac{4x\sin(2x)-\sin^2(2x)}{x^2}$

   xi) $y = \cot\left(\sqrt{(1+x^2)}\right) \quad \Rightarrow y' = \dfrac{-x}{\sqrt{(1+x^2)}}\operatorname{cosec}^2\sqrt{(1+x^2)}$

   xii) $y = (\sec(x)+\tan(x))^3 \quad \Rightarrow y = 3\sec(x)\{\sec(x)+\tan(x)\}^3$

   xiii) $y = x\operatorname{cosec}(x) \quad \Rightarrow y = \operatorname{cosec}(x)\{1-x\cot(x)\}$

3. Note for author : Mainly taken from Linzhong Li (UCL exam A002 2003)

i) $y = \{\cos(x)\}^{1/2} \Rightarrow y' = -\dfrac{\sin(x)}{2(\cos(x))^{1/2}}$

ii) $y = x^2 \ln(x) \Rightarrow y' = x(2\ln(x) + 1)$

iii) $y = \{1 + 5x^3\}^{-2/3} \Rightarrow y' = -10x^2(1 + 5x^3)^{-5/3}$

iv) $y = \ln(3x^2) \Rightarrow y' = \dfrac{2}{x}$

v) $y = x^3 e^{(3x+2)} \Rightarrow y' = 3x^2(1 + x)e^{(3x+2)}$

vi) $y = e^x \sin(x) \Rightarrow y' = e^x(\sin(x) + \cos(x))$

vii) $y = e^{\cos(x)} \Rightarrow y' = -\sin x\, e^{\cos x}$

viii) Take logs of both sides $\Rightarrow$ $y' = x^x\{1 + \ln x\}$

4. Surface area , $A = 6x^2 + 20x + 8 \Rightarrow \dfrac{dA}{dx} = 12x + 20$

5. If we denote the large sheet as having length $2a$ then the cut-out squares must have length $a/3$.

6. $x^4 = a/b$. Compare your answer with Exercise 4.5 Question 13.

7. $\dfrac{dy}{dx} = x^x\{1 + \ln(x)\}$

Exercise 16.5 Solutions

1. i) $y = \text{Sin}^{-1}\left(\frac{x}{a}\right) \Rightarrow y' = \frac{1}{\sqrt{(a^2 - x^2)}}, \quad a > |x|$

ii) $y = \text{Cos}^{-1}\left(\frac{x}{a}\right) \Rightarrow y' = \frac{-1}{\sqrt{(a^2 - x^2)}}, \quad 0 < y < \pi$

iii) $y = \text{Tan}^{-1}\left(\frac{x}{a}\right) \Rightarrow y' = \frac{a}{a^2 - x^2}, \quad -\pi/2 < y < \pi/2$

iv) $y = \sinh^{-1}\left(\frac{x}{a}\right) \Rightarrow y' = \frac{1}{\sqrt{(a^2 + x^2)}}, \quad \text{all } y$

v) $y = \text{Cosh}^{-1}\left(\frac{x}{a}\right) \Rightarrow y' = \frac{1}{\sqrt{(x^2 - a^2)}}, \quad x > a, y > 0$

vi) $y = \tanh^{-1}\left(\frac{x}{a}\right) \Rightarrow y' = \frac{a}{a^2 - x^2}, \quad x^2 < a^2$

vii) $y = \coth^{-1}\left(\frac{x}{a}\right) \Rightarrow y' = \frac{-a}{x^2 - a^2}, \quad x^2 > a^2$

Exercise 16.6 Solutions

1. Leibniz's formula where the coefficients are binomial coefficients.

2. i) $\frac{d^3}{dx^3}\left(x^6\sin(x)\right) = \cos(x)\left\{90x^4 - x^6\right\} + \sin(x)\left\{120x^3 - 18x^5\right\}$

   ii) $\frac{d^4}{dx^4}\left(x^2\cos(2x)\right) = \cos(2x)\left\{16x^2 - 48\right\} + 64x\sin(2x)$

   iii) $\frac{d^2}{dx^2}\left(x^3\tan(x)\right) = 2x^3\sec^2(x)\tan(x) + 6x^2\sec^2(x) + 6x\tan(x)$

3. Differentiate $y = e^{-x}\cos(x)$ four times and plug it in $\frac{d^4y}{dx^4} + 4y = 0.$

4. $\frac{dy}{dx} = -\frac{3x+y}{x+4y}$

5. i) $\frac{dy}{dx} = \frac{e^{x+y} - 2x}{2y - e^{x+y}}$

   ii) $\frac{dy}{dx} = -\frac{y}{\alpha x}$

   iii) $\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$

   iv) $\frac{dy}{dx} = -\frac{y^2\cos\left(xy^2\right) + 2xy\sin\left(x^2y\right)}{2xy\cos\left(xy^2\right) + x^2\sin\left(x^2y\right)}$

6. i) $\frac{dy}{dx} = -\cot(\theta)$

   ii) $\frac{dy}{dx} = \frac{\sin\left(\frac{at}{b}\right) - \sin(t)}{1 - \cos\left(\frac{at}{b}\right)}$

7. Requires differentiating twice and then manipulating the expression. Finally compare coefficients of $\cosh(x)$, to give $A = -1, \quad B = -\frac{1}{2}$.

Exercise 16.7 Solutions

1. i) Use the formula $\cos X + \cos Y = 2\cos\frac{1}{2}(X+Y)\cos\frac{1}{2}(X-Y)$

   ii) $T_0 = 1, \quad T_1 = x, \quad T_2 = 2x^2 - 1, \quad T_3 = 4x^3 - 3x$

   iii) Propose $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ and prove this induction

   iv) The leading coefficient is 2^{n-1}. The constant term is 1, 0, -1, 0, 1,.... the same as $\cos\frac{n\pi}{2}$ while the coefficient of the x is given by $n\sin\frac{n\pi}{2}$

Exercise 17.1 Solutions

1. $(1+x)^n = 1 + nx + \frac{x^2}{2!}n(n-1) + ...$

2. No series expansion for ln(x) since $f(0)$, $f'(0)$ do not exist.

3. Just differentiate $f(x) = \ln(1+x)$ and plug the results into the Maclarin series.

4. Use the series representation for the two exponential terms, multiply together and then ignore terms of x^2 and higher.

5. First 3 tems gives 9 decimal place accuracy so very fast convergence.

6. Very slow convergence. To get 4 decimal place accuracy we need 1 million terms!

7. Much better convergence. First 4 terms give 4 decimal place accuracy.

8. Differentiating $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ gives $\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$. We can multiply by x to give $\sum_{k=0}^{\infty} k\left(\frac{1}{3}\right)^k = \frac{3}{4}$

9. This is L'Hopital's rule $f(x) \approx f(a) + (x-a)f'(a)$, the same for $g(x)$, with both $f(a) = 0$ and $g(a) = 0$ then the $(x-a)$ term cancels to leave $\lim_{x \to a}\left(\frac{f'(x)}{g'(x)}\right)$

10. i) The limit is a $\frac{0}{0}$ so $L = \lim_{x\to 1} \frac{1-x^{n+1}}{1-x} = \lim_{x\to 1} \frac{\frac{d}{dx}\left(1-x^{n+1}\right)}{\frac{d}{dx}(1-x)} = \frac{-(n+1)x^n}{-1} = n+1$

Recall for the Geometric series $\sum_{k=1}^{n} k^n = \frac{\left(1-x^{n+1}\right)}{1-x}$, which is not valid for $x=1$,

but with $x=1$ then the series is just $1+1+1^2+\ldots. \quad +1^n = n+1$

ii) $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$

Authors note: The following questions have been strongly motivated by problem sheets set by Keith Ball.

11. i) Straightforward $c = \frac{b+a}{2}$

ii) Not so straightforward, *c* is given by $c = \pm\sqrt{\frac{b^2+ab+a^2}{3}}$ but now not so easy to show that this lies between *a* and *b*, but do-able.

Both these are examples of the Mean Value Theorem which in turn relies on Rolle's theorem for continuous functions.

12. Use the mean value theorem with $f(x) = \ln(1+x)$ and $a = 0 \Rightarrow f(a) = \ln(1) = 0$

13. i) $f(x) = \frac{1}{3-x} = \frac{1}{3}\left\{1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^3 + \ldots\right\}$ near $x=0$

ii) $f(x) = e^x = e^3\left\{1 + (x-3) + \frac{(x-3)^2}{2!} + \ldots\right\}$ near $x=3$

iii) $f(x) = \ln x = (x-1) - \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \ldots$ near $x=1$

Now we have this put $x = \frac{1}{2}$ and truncate after say 3 terms

iv) $f(x) = \ln x = \ln 4 + (x-4)\frac{1}{4} - \frac{(x-4)^2}{2!}\frac{1}{4^2} + \frac{(x-4)^3}{3!}\frac{1}{4^3} - \ldots$ near $x=4$

14. i) $f(x)=\dfrac{1}{1-x}=1+x+x^2+x^3+....$

ii) Use the result for part (i) with $y=x+2x^2$ to show

$$f(x)=\frac{1}{1-x-2x^2}=1+x+3x^2+5x^3+....$$

iii) Again use earlier results $f(x)=\dfrac{x}{1-x-2x^2}=x+x^2+3x^3+5x^4+....$

iv) $f(x)=\left(\dfrac{1}{1-x-2x^2}\right)^2=1+2x+7x^2+16x^3+....$

Exercise 18.1 Solutions

1. Standard integrals. Note that usually we are using natural logs (ln) but sometimes older teachers (such as myself) write log on the board

i) $\int \sin(x)\,dx = -\cos(x)$

ii) $\int \cos(x)\,dx = \sin(x)$

iii) $\int \tan(x)\,dx = -\ln|\cos(x)|$

iv) $\int \cot(x)\,dx = \ln|\sin(x)|$

v) $\int \sec(x)\,dx = \ln|\sec(x)+\tan(x)|$

vi) $\int \operatorname{cosec}(x)\,dx = \ln\left|\tan\left(\frac{x}{2}\right)\right|$

vii) $\int \sec^2(x)\,dx = \tan^2(x)$

viii) $\int \operatorname{cosec}^2(x)\,dx = -\cot(x)$

ix) $\int \sec(x)\tan(x)\,dx = \sec(x)$

x) $\int \operatorname{cosec}(x)\cot(x)\,dx = -\operatorname{cosec}(x)$

xi) $\int \sinh(x)\,dx = \cosh(x)$

xiii) $\int \cosh(x)\,dx = \sinh(x)$

xiv) $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\mathrm{Tan}^{-1}\left(\frac{x}{a}\right)$

xv) $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|,\quad a>0 \text{ and } a \neq x$

xvi) $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|,\quad a>0 \text{ and } a \neq x$

xvii) $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \mathrm{Sin}^{-1}\left(\frac{x}{a}\right)$

xviii) $\int \frac{1}{\sqrt{x^2+k}}\,dx = \ln\left|x+\sqrt{(x^2+k)}\right|$

2. For authors benefit: Taken from Linzhong Li (UCL exam A002 2003)

i) $\int_0^1 x^2 e^{-x^3}\,dx = \frac{1}{3}\left(1-e^{-1}\right)$

ii) $\int_e^3 \frac{1}{x\ln(x)}\,dx = \ln(\ln(3))$

iii) $\int_1^3 \frac{x^4+1}{x^2}\,dx = \frac{28}{3}$

iv) $\int_0^{\pi/2} \cos^3(x)\,dx = \frac{2}{3}$

Exercise 18.2 Solutions

Do not forget to include the arbitrary constant, c. Most importantly you can check the answers by differentiating your solution.

1. i) $\int \sinh(3x-1)dx = \frac{1}{3}\cosh(3x-1)+c$

ii) $\int e^{1-2x}dx = -\frac{1}{2}e^{1-2x}+c$

iii) $\int 2\tan\left(\frac{x}{2}\right)dx = -4\ln\left|\cos\left(\frac{x}{2}\right)\right|+c$

iv) $\int \sin(x)\cos^3(x)dx = -\frac{1}{4}\cos^4(x)+c$

v) $\int x(3-2x)^4 dx = -\frac{1}{4}\left\{\frac{3}{5}(3-2x)^5 - \frac{1}{6}(3-2x)^6\right\}+c$

vi) $\int x(3-2x^2)^4 dx = -\frac{1}{20}(3-2x^2)^5 = c$

2. i) $\int \frac{1}{\sqrt{(9-x^2)}}dx = \mathrm{Sin}^{-1}\left(\frac{x}{3}\right)+c$

ii) $\int \frac{1}{49+x^2}dx = \frac{1}{7}\mathrm{Tan}^{-1}\left(\frac{x}{7}\right)+c$

iii) $\int \frac{1}{16+9x^2}dx = \frac{1}{12}\mathrm{Tan}^{-1}\left(\frac{3x}{4}\right)+c$

iv) $\int \frac{1}{\sqrt{(9-4x^2)}}dx = \frac{1}{2}\mathrm{Sin}^{-1}\left(\frac{2x}{3}\right)+c$

3. i) $\int e^{3x}dx = \frac{1}{3}e^{3x} + c$

ii) $\int \frac{x}{\sqrt{(1+x^2)}}dx = \sqrt{(1+x^2)} + c$

iii) $\int \frac{x}{1+x^2}dx = \frac{1}{2}\ln(1+x^2) + c = \frac{1}{2}\ln(k(1+x^2)), \quad c = \ln k$

iv) $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln(c|ax+b|)$

v) $\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, \quad a \neq 0, n \neq -1$

vi) $\int \sin(ax+b)dx = \frac{1}{a}\cos(ax+b) + c$

vii) $\int \tan(x)dx = -\ln(k|\cos(x)|)$

viii) $\int \cot(x)dx = \ln(k|\sin(x)|)$

ix) $\int \tanh(x)dx = \ln(k\cosh(x))$

x) $\int \coth(x)dx = \ln(k|\sinh(x)|)$

xi) $\int \frac{1}{a^2\sin^2(x) + b^2\cos^2(x)}dx = \frac{1}{ab}\text{Tan}^{-1}\left(\frac{a}{b}\tan(x)\right) + c$

xii) $\int \frac{1}{\sin(x)}dx = \ln\left(k\left|\tan\left(x/2\right)\right|\right)$

xiii) $\int \frac{1}{\cos(x)}dx = \ln\left(k\left|\tan\left(\frac{x + \pi/2}{2}\right)\right|\right)$

xiv) $\int \cos^2(x)dx = \frac{1}{2}x + \frac{1}{2}\sin(x)\cos(x) + c$

xv) $\int \sin^2(x)dx = \frac{1}{2}x - \frac{1}{2}\sin(x)\cos(x) + c$

xvi) $\int \sqrt{(1-x^2)}dx = -\frac{1}{2}\left\{\cos^{-1}(x) - x\sqrt{(1-x^2)}\right\} + c$

xvii) $\int \sqrt{(a^2-x^2)}dx = -\frac{a^2}{2}\cos^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{(a^2-x^2)} + c$

xviii) Use either $1-x=u$ or $1-x=t^2$, $\int \frac{x}{\sqrt{(1-x)}}dx = -\frac{2}{3}(1-x)^{1/2}(2+x) + c$

xvii) $\int \sqrt{(a^2-x^2)}dx = -\frac{a^2}{2}\cos^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{(a^2-x^2)} + c$

4. i) Use $x = \sin^2\theta \Rightarrow \int \sqrt{\left(\frac{x}{1-x}\right)}dx = \text{Sin}^{-1}\left(\sqrt{x}\right) - \sqrt{x(1-x)} + c$

ii) Use $x = \tan\theta \Rightarrow \int \frac{1}{\left(1+x^2\right)^{3/2}}dx = \frac{x}{\sqrt{\left(1+x^2\right)}} + c$

iii) Use $x = a\cos(2\theta) \Rightarrow \int \sqrt{\left(\frac{a+x}{a-x}\right)}dx = -a\left\{Cos^{-1}\left(\frac{x}{a}\right) + \frac{1}{a}\sqrt{\left(a^2 - x^2\right)}\right\} + c$

iv) Use $x = \sec(x) \Rightarrow \int \frac{1}{x\sqrt{\left(x^2-1\right)}}dx = \text{Sec}^{-1}(x) + c$

5. $\int \frac{1}{4x^2 + 4x + 5}dx = \frac{1}{4}\text{Tan}^{-1}\left(\frac{2x+1}{2}\right) + c$

6. i) $\int \sin^{-1}(x)dx = x\text{Sin}^{-1}(x) + \sqrt{\left(1-x^2\right)} + c$

ii) $\int \tan^{-1}(x)dx = x\,Tan^{-1}(x) - \frac{1}{2}\ln\left(1+x^2\right) + c$

7. Recall that we have said that $\ln(x) = \int_1^x \frac{1}{t}\,dt$ is perfectly good definition for $\ln(x)$.
$\int_1^4 \frac{1}{x}\,dx = \int_1^2 \frac{1}{x}\,dx + \int_2^4 \frac{1}{x}\,dx = 2\int_1^2 \frac{1}{x}\,dx \Rightarrow \ln 4 = 2\ln 2$. This agrees with our definition for the logarithmic function

Exercise 18.3 Solutions

In the indefinite integrals below we could let the arbitrary constant be $c = \ln(k)$ to simplify the expressions.

1. $\displaystyle\int \frac{3x}{x^2 + x - 2}\,dx = \ln\{|(x-1)|(x+2)^2\} + c$

2. i) $\displaystyle\int \frac{x^2 + 2x + 1}{(x-1)((x+2)(x+3)}\,dx = \frac{1}{3}\ln\left|\frac{(x-1)(x+3)^3}{x+2}\right| + c$

ii) $\displaystyle\int_0^2 \frac{4}{(3-x)(x+1)}\,dx = \ln 9 \approx 2.20$

iii) $\displaystyle\int_2^3 \frac{1}{(x^2-1)}\,dx = \frac{1}{2}\ln\frac{3}{2} = \ln\sqrt{\frac{3}{2}} \approx 0.20$

iv) $\displaystyle\int \frac{3x+1}{x^2 + x - 2}\,dx = \frac{1}{3}\ln|(x-1)^4(x+2)^5| + c$

v) $\displaystyle\int \frac{1}{x^2 - 5x + 6}\,dx = \ln\left|\frac{(x-3)}{x-2}\right| + c$

vi) $\displaystyle\int \frac{1}{x(x+1)}\,dx = \ln\left|\frac{x}{x+1}\right| + c$

vii) $\displaystyle\int \frac{3x+2}{x^2 - 2x}\,dx = \ln\left|\frac{(x-2)^2}{x}\right| + c$

vii) $\displaystyle\int \frac{1}{x(x-1)^2}\,dx = \ln\left|\frac{x}{x-1}\right| - \frac{1}{x-1} + c$

ix) $\displaystyle\int \frac{2x}{(x-1)^2(x+2)}\,dx = \ln\left|\left(\frac{x-1}{x+2}\right)^{4/9}\right| - \frac{2}{3(x-1)} + c$

x) $\displaystyle\int \frac{4x}{(x^2+4)(x+1)}\,dx = \frac{2}{5}\ln\left|\frac{x^2+4}{x+1}\right| + \frac{8}{5}\mathrm{Tan}^{-1}\left(\frac{x}{2}\right) + c$

xi) $\displaystyle\int \frac{x^2 + 4x + 5}{x^2 + 4x + 13}\,dx = x + \ln\left|\frac{x+1}{x+3}\right| + c$

xii) $\displaystyle\int \frac{x^2 + 4x + 5}{(x+2)^2}\,dx = x + 2\ln|x+2| - \frac{3}{x+2} + c$

Exercise 18.4 Solutions

1. i) $$\int x(3-2x)^4 dx = -\frac{1}{10}x(3-2x)^5 - \frac{1}{120}(3-2x)^6 + c.$$

This does not initially look the same as the answer

$$-\frac{1}{4}\left\{\frac{3}{5}(3-2x)^5 - \frac{1}{6}(3-2x)^6\right\} + c$$

from Exercise 18.2 Question 1 (v) but it can be rearranged to be the same.

ii) $$\int_0^1 xe^x dx = 1$$

iii) $$\int \ln(x)dx = x\ln(x) - x + c$$

iv) $$\int x\sin(x)dx = -x\cos(x) + \sin(x) + c$$

v) $$\int x\cos(x)dx = x\sin(x) + \cos(x) + c$$

vi) $$\int \frac{\ln(x)}{x}dx = \frac{1}{2}(\ln(x))^2 + c$$

vii) $$\int x^a \ln(x)dx = \frac{x^{a+1}}{a+1}\left\{\ln(x) - \frac{1}{a+1}\right\}, \quad a \neq 1$$

viii) $$\int \frac{\ln(1+x)}{x^2}dx = \ln(x) - \frac{1+x}{x}\ln(1+x) + c$$

ix) $$\int \ln\left(\frac{1}{x}\right)dx = x(1-\ln(x)) + c$$

x) $$\int x^2 e^{3x}dx = \frac{e^{3x}}{27}\{9x^2 - 6x + 2\} + c$$

xi) Use integration by parts twice $\int e^x \cos x\, dx = \frac{1}{2}e^x(\cos x + \sin x)$

xii) $$\int e^{4x}\cos(3x)dx = \frac{1}{25}e^{4x}\{3\sin(3x) + 4\cos(3x)\} + c$$

xiii) Use the substitution $y = \ln(x)$ and the integral becomes $\int y^2 e^y\, dy$ which we then do by parts, twice, to give

$$\int (\ln(x))^2 dx = x(\ln(x))^2 - 2x\ln(x) + 2x + c$$

Remember it is reasonably easy to check your answer by differentiating.

Exercise 18.5 Solutions

1. i) $I_n = \int (\ln(x))^n \, dx = x(\ln(x))^n - nI_{n-1}$ Check that it works for $n = 0, 1$

ii) $I_n = \int \tan^n(x) dx = \frac{1}{n-1}(\tan(x))^{n-1} - I_{n-2}$ and $\int_0^{\pi/4} \tan^5(x) dx = \ln(\sqrt{2}) - 1/4$

iii) $I_n = \int_0^\infty x^n e^{-x} dx = nI_{n-1}$ and hence $I_n = n!$

iv) Integrating by parts gives

$$I_n = \int (1-x^3)^n \, dx = x(1-x^3)^n + 3n\int x^3(1-x^3)^{n-1} dx.$$

Writing $x^3 = 1-(1-x^3)$ leads to $(3n+1)I_n = x(1-x^3)^n + 3nI_{n-1}$

When the limits are 0 and 1 the first term is zero. Calculating $I_1 = \frac{3}{4}$ gives the result.

v) $nI_n = \sin(x)\cos^{n-1}(x) + (n-1)I_{n-2}$

The first term vanishes at both limits so $I_n = \frac{n-1}{n} I_{n-2}$

Thus different result for n odd or n even. Note $I_0 = \pi/2$, $I_1 = 1$

Exercise 18.6 Solutions

1. Either using geometric arguments, or consider $\int_{-L}^{L} f(x)dx = \int_{-L}^{0} f(x)dx + \int_{0}^{L} f(x)dx$ and use the substitution $u = -x$ in the first integral.

2. Same approach as in Question 1.

3. Depends on the limits. Think of $\int_{-L}^{L} \cos(x)dx$, if $L = \pi$ then integral is zero but not for $L = \pi/2$, for instance.

4. Either an integral of an odd function between symmetric limits is zero, or consider

$$\sin A \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}.$$

5. More general form than Qu 4 used in Fourier series.

   i) $\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = 0$ since we have the integral of an odd function between symmetric limits

   ii) $\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$ may, or may not be zero since integrand is an even function. Use $\sin A \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}$ to show integral is zero. Same for $\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = 0, \quad m \neq n$

   iii) $\int_{-L}^{L} \sin^2\left(\frac{m\pi x}{L}\right)dx = \int_{-L}^{L} \cos^2\left(\frac{m\pi x}{L}\right)dx = L$. Use $1 - \cos 2A = 2\sin^2 A$ for first integral and $1 + \cos 2A = 2\cos^2 A$ for the second.

Exercise 18.7 Solutions

1. To estimate $\int_0^1 e^{x^2}\,dx$ use $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ replacing x with x^2 then integrate the series term by term to give a series between the limits. To give 4 decimal place accuracy we will need to take sufficient terms such that the terms neglected do not change the fourth decimal place. In this case up to $\frac{x^{15}}{15 \times 7!}$ to give $\int_0^1 e^{x^2}\,dx \approx 1.4627$

2. To evaluate $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$ try the substitution $x = \frac{\pi}{2} - y$. This gives a similar integral $\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$ so we can add both together to give $2I = \int_0^{\pi/2} dx = \frac{\pi}{2}$ so $I = \frac{\pi}{4}$

3. Complete the square and use the substitution $1 - x = \sqrt{3}\sin\theta$, $\int_0^1 \frac{1}{\sqrt{(2 + 2x - x^2)}}\,dx = \sin^{-1}\frac{1}{\sqrt{3}}$

4. To evaluate $I = \int_1^2 \frac{1}{(x^2 - 2x + 4)^{3/2}}\,dx$ complete the square in the bottom $x^2 - 2x + 4 = (x-1)^2 + 3$ then use the substitution $x - 1 = \sqrt{3}\tan u$, to give $I = \frac{1}{6}$.

5. Use the substitution $x = \ln t$, $\int \frac{1}{t\ln t}\,dt = \ln(\ln t) + c$

6. $p(x) = \frac{1}{6}\left\{f(1) + 4f\left(\frac{1}{2}\right) + f(0)\right\}$ which is Simpson's rule for approximate integration.

5. Let $L = \lim_{n\to\infty}\left\{\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots \frac{1}{n+n}\right\}$, each term looks to tend to 0. In fact we can re-write as $\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\frac{1}{1+k/n}$. Now recall the definition of an integral in terms of an area, $\int_a^b f(x)\,dx \approx \sum_{k=0}^{n} f(x_k)\delta x$ but $\delta x = \frac{b-a}{n}$ so

$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=0}^{n} f(x_k)\frac{b-a}{n} = \lim_{n\to\infty} \frac{b-a}{n}\sum_{k=0}^{n} f\left(a + k\left(\frac{b-a}{n}\right)\right)$. With $a=1$ and $b=2$ then $f(k/n) = \frac{1}{1+k/n} \Rightarrow L = \int_1^2 \frac{1}{x}dx = \ln(2) - \ln(1) = \ln(2)$

6. Show that $\sum_{k=1}^{\infty} \frac{1}{k} > \lim_{n\to\infty} \int_1^{n+1} \frac{1}{x}dx = \ln(n+1) \to \infty$ hence the series diverges.

7. Note $\int_n^{\infty} \frac{1}{x^2}\,dx = \frac{1}{n}$ and form the partial sums

$$\sum_{k=2}^{\infty} \frac{1}{k^2} \approx 1 + \qquad + \frac{1}{2}$$
$$\sum_{k=2}^{\infty} \frac{1}{k^2} \approx 1 + \frac{1}{4} \qquad + \frac{1}{3}$$
$$\sum_{k=2}^{\infty} \frac{1}{k^2} \approx 1 + \frac{1}{4} + \frac{1}{9} \qquad + \frac{1}{4}$$

etc

8. i) Most calculators only have digits for the exponent and so the most we can calculate is 69!

ii) $\sum_{k=1}^{n} \ln k = \ln(n!) \approx \int_1^n \ln(x)\,dx$. Use this to give $n! \approx n^n e^{-n} e$

iii) Now we get $n! \approx e^{-n}\sqrt{2}\left(n + \frac{1}{n}\right)^{n+1/2}$. According to Keith Ball's notes this is roughly $4\cdot 023 \times 10^{2567}$ when $n = 1000$

Exercise 18.8 Solutions

Proof that π is irrational : You may need some hints. There are several ways to prove that π is irrational. None are particularly easy. The proof that π is transcendental is apparently longer, but this is left as an exercise! The original proof was apparently completed by J.H. Lambert in the 1760's but the following can be seen in many texts books on *Analysis*, for instance see the book by Brannan (2006).

Step 1. Show that $\int_{-1}^{1}\cos\left(\frac{\pi}{2}x\right)dx=\frac{4}{\pi}$ and $\int_{-1}^{1}(1-x^2)\cos\left(\frac{\pi}{2}x\right)dx=\frac{32}{\pi^3}$

Step 2. If $I_n=\int_{-1}^{1}(1-x^2)^n\cos\left(\frac{\pi}{2}x\right)dx$ for $n=1,2,3,....$, then show that

i) $I_n>0$

ii) $I_n\le 2$

Step 3. Using the definition of I_n in Step 2, show that I_n satisfies the relationship

$$\pi^2 I_n=8n(n-1)I_{n-1}-16n(n-1)I_{n-2}$$

Step 4. Define $J_n=\frac{\pi^{2n+1}}{n!}I_n$, then prove, by induction, that there exists integers $a_0, a_1, ... , a_n$ such that $J_n=a_0+a_1\pi+a_2\pi^2+.....+a_n\pi^n$

Step 5. Using the result that $J_n=\sum_{k=0}^{n}a_k\pi^k$, and the further results from the Steps above, prove, by contradiction, that π is irrational.

An alternative approach is as follows

Step 1. Let $f(x) = \dfrac{x^n(1-x^2)^n}{n!}$ and show that $0 < f(x) < \dfrac{1}{n!}$

Step 2. Denote the kth derivative of $f(x)$ as $f^k(x)$. Again we proceed by assuming that π is rational which implies that $\pi^2 = \dfrac{p}{q}$ for some integers p and q. Consider the function

$$F_n(x) = q^n\left\{\pi^{2n} f(x) - \pi^{2n-2} f^2(x) + \pi^{2n-4} f^4(x) - .... + (-1)^n f^{2n}(x)\right\}$$

and show that $F_n(0)$ and $F_n(1)$ are integers.

Step 3. Show that $\dfrac{d}{dx}\{F_n'(x)\sin(\pi x) - \pi F_n(x)\cos(\pi x)\} = p^n\pi^2 \sin(\pi x) f(x)$ and hence show that $\int_0^1 p^n\pi^2 \sin(\pi x) f(x) dx = F_n(1) + F_n(0)$, which is an integer.

Step 4. Starting with the result from Step 1 that $0 < f(x) < \dfrac{1}{n!}$, multiply by $\pi p^n \sin(\pi x)$ and integrate from 0 to 1, to show that this assumptions leads to the the contradiction that an integer lies between 0 and 1.

Exercise 19.1 Solutions

1. i) $z=1 \Rightarrow |z|=1 \quad \text{Arg}(z)=0$ ii) $z=i \Rightarrow |z|=1 \quad \text{Arg}(z)=\pi/2$

iii) $z=-1 \Rightarrow |z|=1 \quad \text{Arg}(z)=\pi$ iv) $z=-i \Rightarrow |z|=1 \quad \text{Arg}(z)=-\pi/2$

v) $z=-2\sqrt{3}-2i \Rightarrow |z|=4 \quad \text{Arg}(z)=-5\pi/6$

vi) $z=\frac{\sqrt{3}}{2}-\frac{3}{2}i \Rightarrow |z|=\sqrt{3} \quad \text{Arg}(z)=-\pi/3$

vii) $z=-\frac{1}{2} \Rightarrow |z|=\frac{1}{2} \quad \text{Arg}(z)=\pi$

viii) $z=-5i \Rightarrow |z|=5 \quad \text{Arg}(z)=-\pi/2$

ix) $z=2+2i \Rightarrow |z|=2\sqrt{2} \quad \text{Arg}(z)=\pi/4$

x) $z=1+\sqrt{3}i \Rightarrow |z|=2 \quad \text{Arg}(z)=\pi/3$

xi) $z=-1-i \Rightarrow |z|=\sqrt{2} \quad \text{Arg}(z)=3\pi/4$

xii) $z=-\sqrt{3}-\frac{1}{\sqrt{3}}i \Rightarrow |z|=\sqrt{\frac{10}{3}} \quad \text{Arg}(z)=-\pi+\tan^{-1}\left(\frac{1}{3}\right)\approx -2.82$

xiii) $z=-\sqrt{3}-i \Rightarrow |z|=2 \quad \text{Arg}(z)=-\pi/6$

xiv) $z=1-i \Rightarrow |z|=\sqrt{2} \quad \text{Arg}(z)=-\pi/4$

xv) $z=-\sqrt{3}+i \Rightarrow |z|=2 \quad \text{Arg}(z)=5\pi/6$

2. Write $\frac{1+i}{2+ai}+\frac{2+3i}{3+i}$ as $x+iy$ then $x=y \quad \Rightarrow a=-5\pm\sqrt{21}$.

3. i) $\arg(z)=\text{Arg}(z)+2k\pi, \quad k=0,\pm 1,\pm 2,....$ is true since the tangent function is periodic with period 2π

ii) $\text{Arg}(\bar{z})=-\text{Arg}(z)$ true by definition of the conjugate use $z=re^{i\theta}$

iii) $\text{Arg}\left(\frac{z_1}{z_2}\right)=\text{Arg}(z_1)-\text{Arg}(z_2)$ true use $z=re^{i\theta}$

iv) $\text{Arg}(z_1z_2)=\text{Arg}(z_1)+\text{Arg}(z_2)$ true use $z=re^{i\theta}$

v) $\text{Arg}\left(\frac{z_1}{z_2}\right)=\text{Arg}(z_1z_2)$ true put $z=re^{i\theta}$ much easier than in Exercise 10.1

4. i) $|z-3+2i|=|z+3i|$ and $\operatorname{Arg}(z)=\pi/4 \Rightarrow z=\frac{1}{2}+\frac{1}{2}i$

ii) $|z+1-2i|=|2z+i|$ and $\operatorname{Arg}(z)=-\pi/4 \Rightarrow z=2-2i$

Exercise 19.2 Solutions

1. i) $z = -2\sqrt{3} - 2i \quad \Rightarrow z = 4e^{i7\pi/6}$

ii) $z = \frac{\sqrt{3}}{2} - \frac{3}{2}i \quad \Rightarrow z = \sqrt{3}e^{i5\pi/3}$

iii) $z = -\frac{1}{2} \quad \Rightarrow z = \frac{1}{2}e^{i\pi}$

iv) $z = -5i \quad \Rightarrow z = 5e^{i3\pi/2}$

v) $z = 2 + 2i \quad \Rightarrow z = 2\sqrt{2}e^{i\pi/4}$

vi) $z = 1 + \sqrt{3}i \quad \Rightarrow z = 2e^{i\pi/3}$

vii) $z = -1 + i \quad \Rightarrow z = \sqrt{2}e^{i3\pi/4}$

viii) $z = \sqrt{3} - \frac{i}{\sqrt{3}} \quad \Rightarrow z \approx \sqrt{\frac{10}{3}}e^{i(\pi + 0.322)}$

ix) $z = \sqrt{3} - i \quad \Rightarrow z = 2e^{i11\pi/6}$

x) $z = 2(1+i) \quad \Rightarrow z = 2\sqrt{2}e^{i\pi/4}$

2. i) $z = 2\left\{\cos\left(3\pi/4\right) + i\sin\left(3\pi/4\right)\right\} \quad \Rightarrow z = -\sqrt{2} + i\sqrt{2}$

ii) $z = 4e^{i5\pi/4} \quad \Rightarrow z = -2\sqrt{2} - i2\sqrt{2}$

iii) $z = 7e^{i7\pi/6} \quad \Rightarrow z = -\frac{7\sqrt{3}}{2} + i\frac{7}{2}$

iv) $z = e^{i5\pi/2} \quad \Rightarrow z = i$

3. Use $z = re^{i\theta}$ to show results.

4. i) $z = i \quad \Rightarrow \ln(z) = i\pi/2$

ii) $z = 1 + i \quad \Rightarrow \ln(z) = \frac{1}{2}\ln(2) + i\pi/4$

5. $\ln(z_1 z_2) = \ln(z_1) + \ln(z_2)$

6. For two complex numbers to be equal their modulus must be equal and their argument must be equal $1 = \sqrt{(-1)}\sqrt{(-1)} = e^{i\pi/2} \cdot e^{-i\pi/2} = i \cdot -i = 1$ is correct.

7. $|z| = 2\cos\left(\frac{\theta}{2}\right)$ and $\arg(z) = \frac{\theta}{2}$

8. Putting $I = \text{Re}\left\{\int e^{(4+3i)x}dx\right\}$ gives $I = \frac{1}{25}e^{4x}\{3\sin(3x) + 4\cos(3x)\} + c$ as in Exercise 18.4 Question 1 part (xii) but this is much easier.

9. You could try using $\cos(\theta + \phi)$ but much easier to say $F(\theta) = e^{i\theta}$ then the result follows from the index laws.

10. Show but remember that $\tan^{-1}(x)$ is an odd function.

11. Comparing real and imaginary parts gives $r = \sqrt{3}$ and $r = \frac{3 \pm \sqrt{5}}{2}$ with $\dot{\theta} = 2$ so anticlockwise in both cases.

Exercise 19.3 Solutions

1. Use formula for $\cos(\theta+\phi)$ (authors note: taken from Keith Ball's notes)

2. Usual induction, but you can use the result of Question 1 (taken from Keith Ball's notes).

3. Use de Moivre's theorem tells us that $(\cos(x)+i\sin(x))^4 = \cos(4x)+i\sin(4x)$. Now multiply out the bracket and compare real and imaginary terms. Much easier than the answer to Exercise 6.2 Question 8.

4. Use de Moivre's theorem, multiply out the bracket and compare real and imaginary terms.

5. Try to show that $|z|=1$. Then for instance for part (iii) multiply the bracket $\cos^3(x) = \left\{\frac{1}{2}\left(z+\frac{1}{z}\right)\right\}^3$ and then put back in terms of $\frac{1}{2}\left(z+\frac{1}{z}\right)$ etc.

6. Start by using de Moivre's theorem to find $\tan(7\theta) = \dfrac{\sin(7\theta)}{\cos(7\theta)}$ in terms of $t = \tan\theta$ then consider when $\tan(7\theta) = 0$ to find an equation whose roots are $\tan^2(\theta)$ (taken from S.L. Green, Theory and use of the complex variable, Pitman, London).

Exercise 19.4 Solutions

1. i) $\sqrt{i} = \pm\frac{1}{2}\sqrt{2}(1+i)$

   ii) $\sqrt{(-5+12i)} = \pm(2+3i)$

   iii) $\sqrt{(-3+4i)} = \pm(1+2i)$

   iv) $\frac{1}{\sqrt{(3-4i)}} = \pm\frac{1}{5}(2+i)$

2. Let $z = 1+i\sqrt{3}$ then $z^{3/4} = 2^{1/4}(1+i),\ 2^{1/4}(1-i),\ 2^{1/4}(-1+i),\ 2^{1/4}(-1-i)$

3. i) $i^{\frac{1}{4}} = e^{i\frac{\pi}{8}}, e^{i\frac{5\pi}{8}}, e^{-i\frac{3\pi}{8}}, e^{-i\frac{7\pi}{8}}$

   ii) $(-1)^{\frac{3}{4}} = e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{-i\frac{\pi}{4}}, e^{-i\frac{3\pi}{4}}$

4. Formula gives $z = \frac{1}{2}\left\{-3-2i \pm 3\sqrt{(-3+4i)}\right\}$. Using Question 1 part (iii) gives $z = 2i, -3-4i$.

5. Let $z = 1 = e^{i2k\pi}$ then with $\theta = 2\pi/n \Rightarrow \sum_{k=0}^{n-1}\omega^k = 1 + e^{i\theta} + e^{2i\theta} + ... + e^{(n-1)i\theta}$ which can be summed as a geometric series. Also note that since $z^5 = 1$ then $z^5 - 1 = (z-1)\left(1+z+z^2+z^3+z^4\right) = 0$ and if $z \neq 1$ this implies that $1+z+z^2+z^3+z^4 = 0$

6. i) $z = -1,\ \frac{1}{2} - i\frac{\sqrt{3}}{2},\ \frac{1}{2} + i\frac{\sqrt{3}}{2}$

   ii) $z = 1\cdot084 + i0\cdot291,\ -0\cdot794 + i0\cdot794,\ -0\cdot291 - i1\cdot084$

   iii) $z = 1\cdot629 - i0\cdot520,\ -0\cdot364 + i1\cdot671,\ -1\cdot265 - i1\cdot151$

7. i) $z^3 = 4\left(1-i\sqrt{3}\right) \Rightarrow z = 2e^{-i\pi/9}, 2e^{-i7\pi/9}, 2e^{i5\pi/9}$

   ii) $z^4 - 2z^3 + 4z^2 - 8z + 16$ is a geometric series whose sum is zero when $z^5 = -32 \Rightarrow z = 2e^{i\pi/5}, 2e^{i3\pi/5}, 2e^{-i\pi/5}, 2e^{-i3\pi/5}$

8. Quite long. More than one way. Use $z = x + iy$ plug it in and use $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$ and then equate real and imaginary parts to show that $x = (2n+1)\pi/2$, $y = \ln(2 \pm \sqrt{3})$. Or use $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$.

9. Use the result $z^5 - 1 = (z-1)(1 + z + z^2 + z^3 + z^4)$ so that $u + v = -1$ and $uv = -1$. We can solve these to give $u = 0 \cdot 618$. Now $z + z^4 = 2\cos\left(\dfrac{2\pi}{5}\right) = -1$ hence result.

10. As for question 8 but this time u and v are complex.

11. $(z+1)^5 + z^5 = 0$ can be re-written as $\left(\dfrac{z+1}{z}\right)^5 = -1$, so let $w = \dfrac{z+1}{z}$ then $w^5 = e^{i(\pi + 2k\pi)}$ to give $z = -0 \cdot 5 - 1 \cdot 539i,\quad -0 \cdot 5 - 0 \cdot 363i,\quad -0 \cdot 5 + 0 \cdot 363i,\quad -0 \cdot 5 + 1 \cdot 539i$

Exercise 20.1 Solutions

1. The idea here is that students just look at the changes in sign of the derivative for values either side of the points for which the gradient is zero

   i) $y = x^2 + 2$ minimum at $x = 0$

   ii) $y = 2x^2 - 4x$ minimum at $x = 1$

   iii) $y = (2x+1)(x-3)$ minimum at $x = 5/4$

   iv) $y = x^3$ has a point of inflexion at $x = 0$

   v) $y = 2x^3 + x^2 - 4x + 1$ minimum at $x = 2/3$, maximum at $x = -1$

   vi) $y = x^3 - 3x$ minimum at $x = 1$, maximum at $x = -1$

   vii) $y = x^4$ minimum at $x = 0$

   viii) $y = x^2(x^2 - 2)$ minimum at $x = 1, -1$, maximum at $x = 0$.

2. Same functions as in Question 1 but now find the second derivatives to confirm the results.

3. Probably best to use value of second derivative

   i) $y = \dfrac{x^3}{x^2 + 3}$ point of inflection at $x = 0$,

   ii) $y = 2x^5 - 9x^4 + 8x^3 + 8x^2 - 16$ minimum at $x = 0$, point of inflection at $x = 2$, maximum at $x = -2/5$

   iii) $y = x^2(x-12)^2$ minimum at $x = 0$ and $x = 12$, maximum at $x = 6$

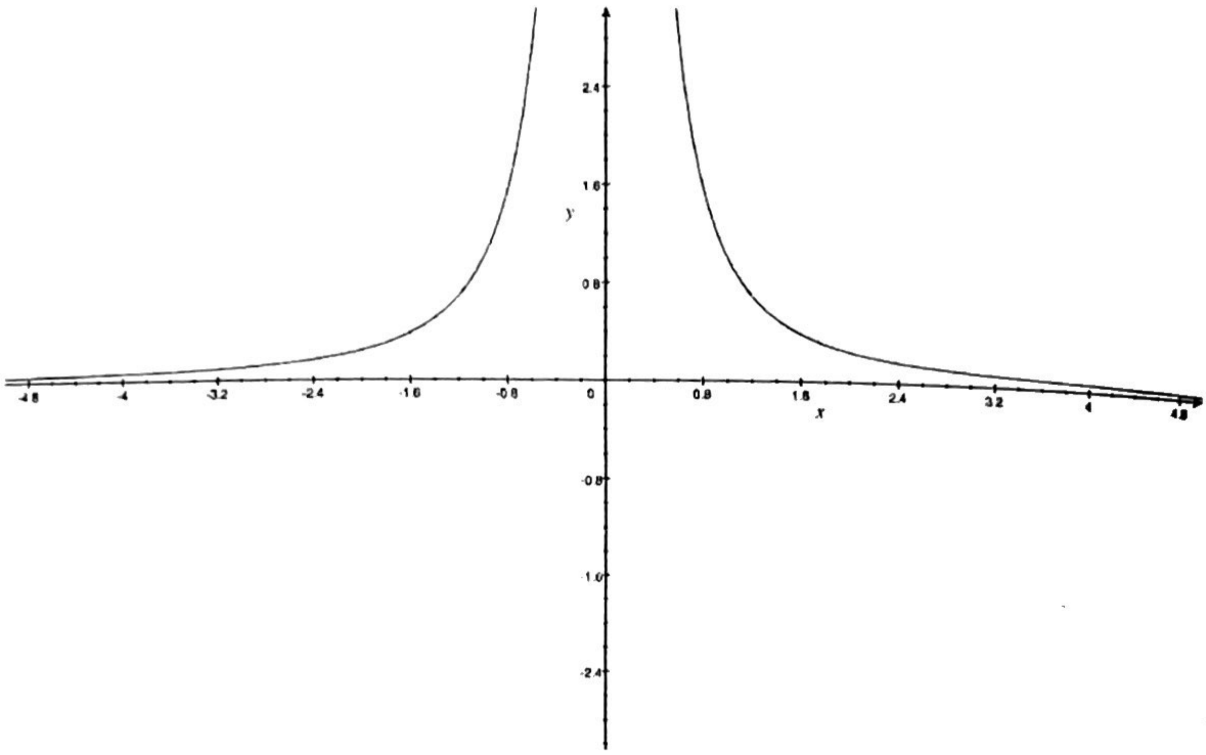
   iv) $y = (x^4 + 5x^2 + 8x + 8)e^{-x}$ minimum at $x = 0$, point of inflection at $x = 1$, maximum at $x = 2$.

4. Factorize y as a polynomial, range to help the student find the solutions, then solve to show minimum at $x = 2$, maximum at $x = 1$

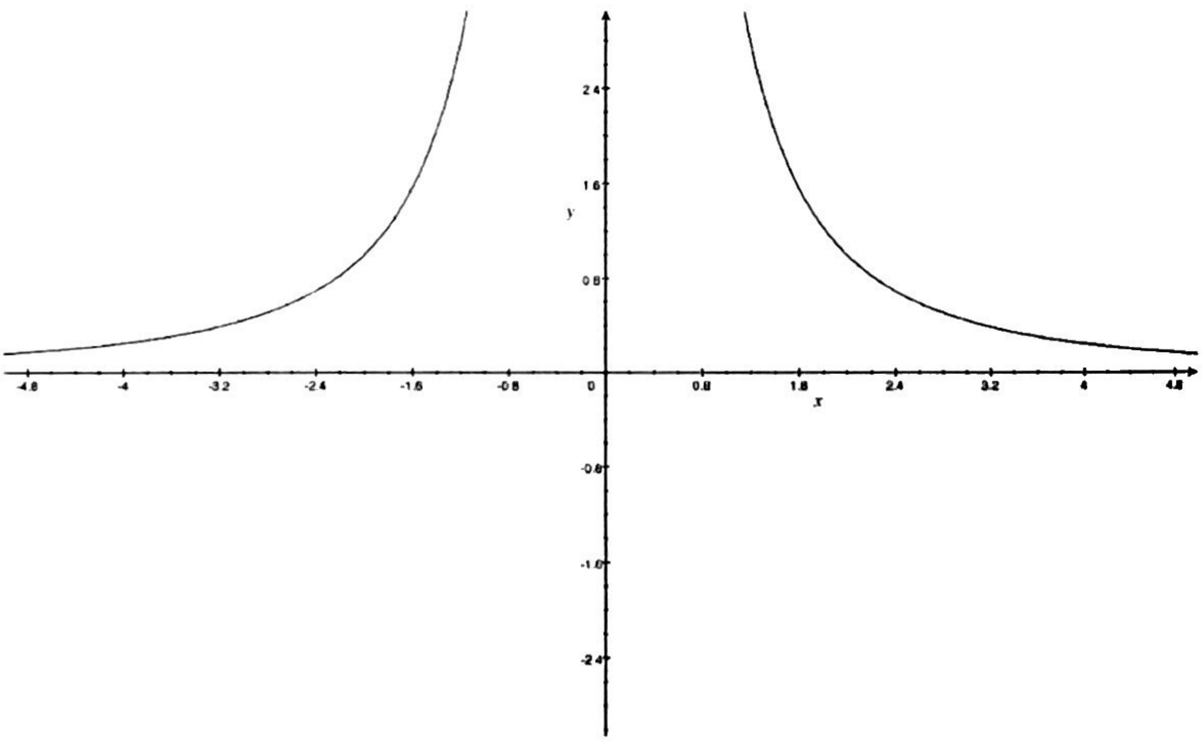
5. You can write the function as an algebraic expression and then find the coefficient of the highest power of y and x. Equate these to zero to find the asymptotes.

i) Basic curve, $y > 0$. Asymptote on y axis, $x = 0$

$$y = \frac{1}{x^2}$$



ii) Same shape as (i) $y = \frac{4}{x^2}$



iii) $y = 2x^2 - 4x + 1.$

Quadratic.

a) $y = 0$ when $x = 0.3, 1.7$, $x = 0$ when $y = 1$ so intercepts at

$(0,1), (0.3,0)$ and $(1.7,0)$

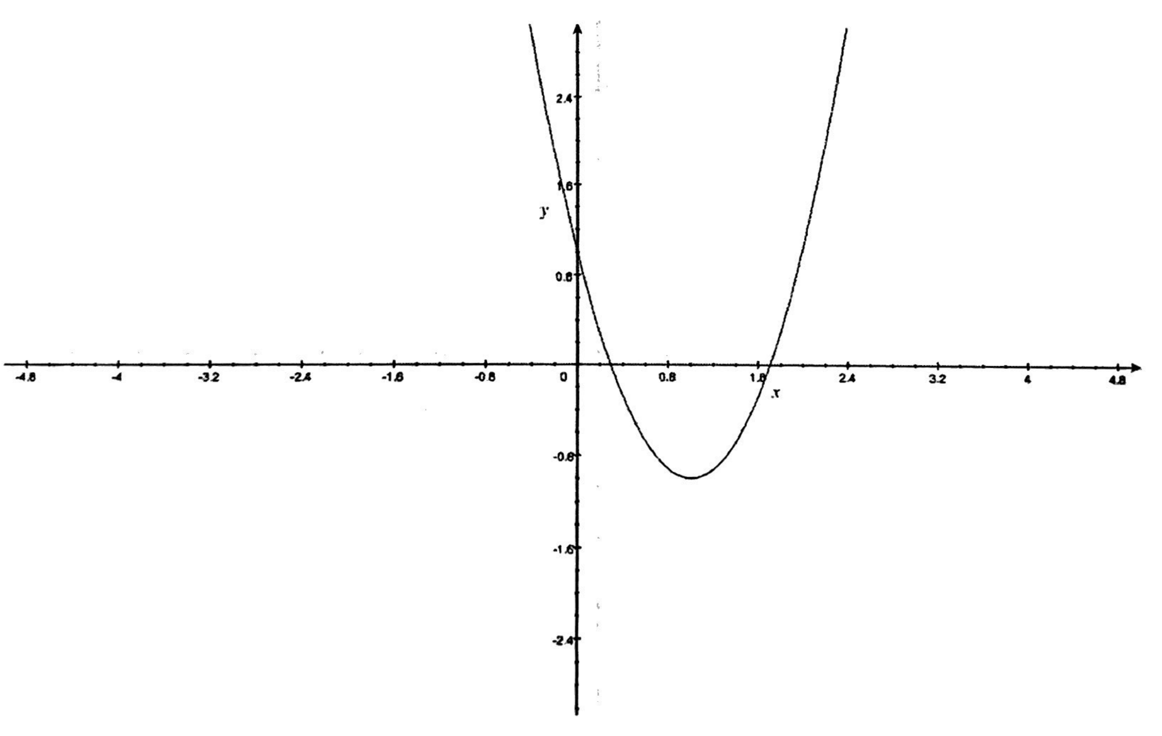
b) No symmetry

c) When x is large, y behaves like $y = 2x^2$. When x is small, y behaves like $y = 1 - 4x$

d) No vertical asymptotes

e) Minimum at $(1,-1)$.

$y = 2x^2 - 4x + 1$



iv) $y = 2 + x^2 - x^3$.

Cubic

a) $y = 0$ when $x \approx 1.7$, $x = 0$ when $y = 2$ so intercepts at (0,2) and (1.7,0)

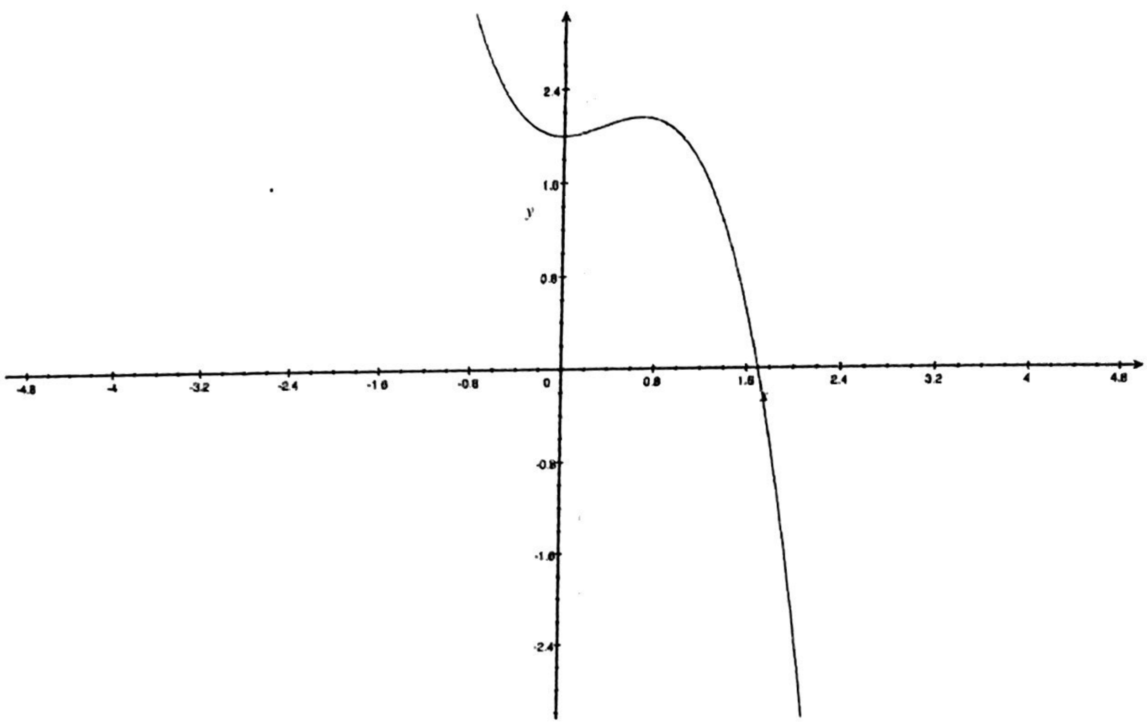
b) No symmetry

c) When x is large, y behaves like $y = -x^3$. When x is small, y behaves like $y = 2$ a straight line

d) No vertical asymptotes

e) Minimum at (0,2), maximum at $\left(\frac{2}{3}, \frac{58}{27}\right)$.

$$y = 2 + x^2 - x^3$$



v) $y = \dfrac{1}{x^2+1}$.

Note $y > 0$

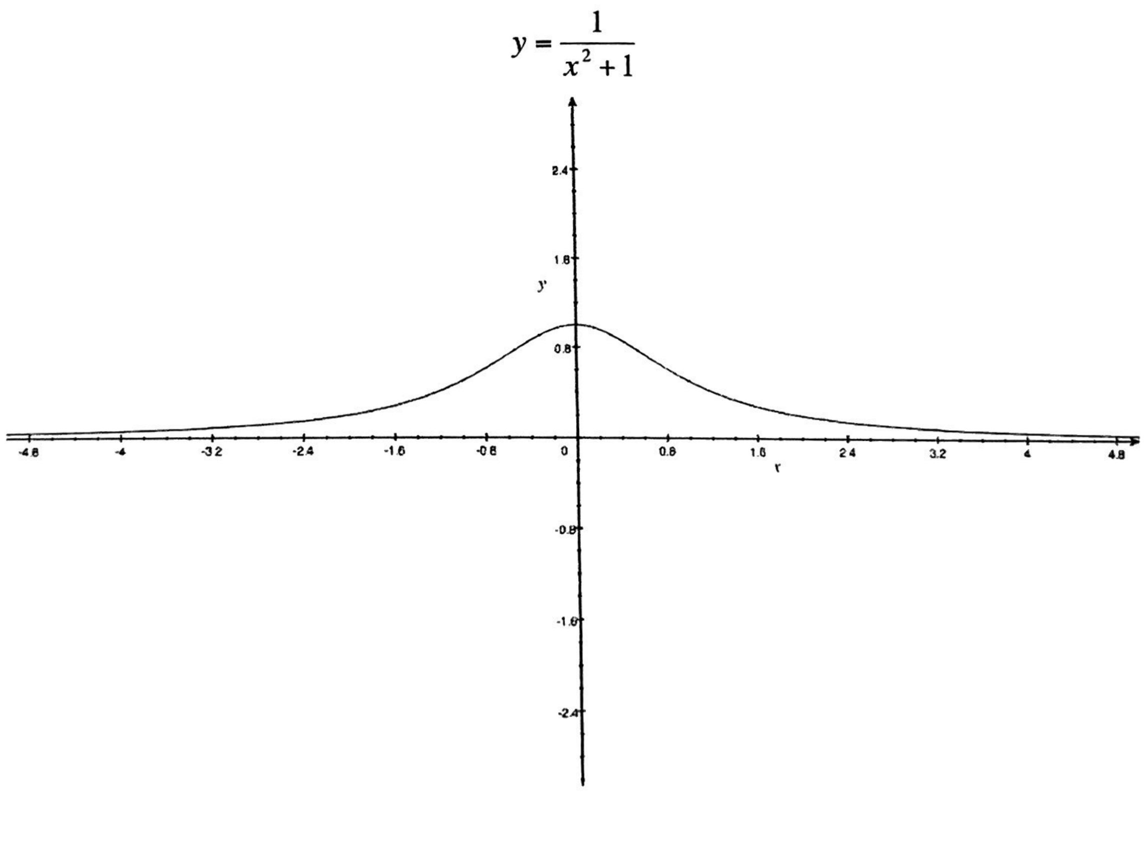
a) $x = 0$ when $y = 1$ so intercepts at $(0,1)$

b) Symmetry about y-axis (replacing x by $-x$ gives the same equation)

c) When x is large, y behaves like $y = 1/x^2$. When x is small, y behaves like $y = 1$ a straight line

d) No vertical asymptotes

e) Maximum at $(0,1)$, points of inflexion at $\left(\pm\sqrt{1/3}, 3/4\right)$.



vi) $y = \dfrac{x^2}{x+2}$.

a) $x = 0$ when $y = 0$ so intercepts at (0,0)

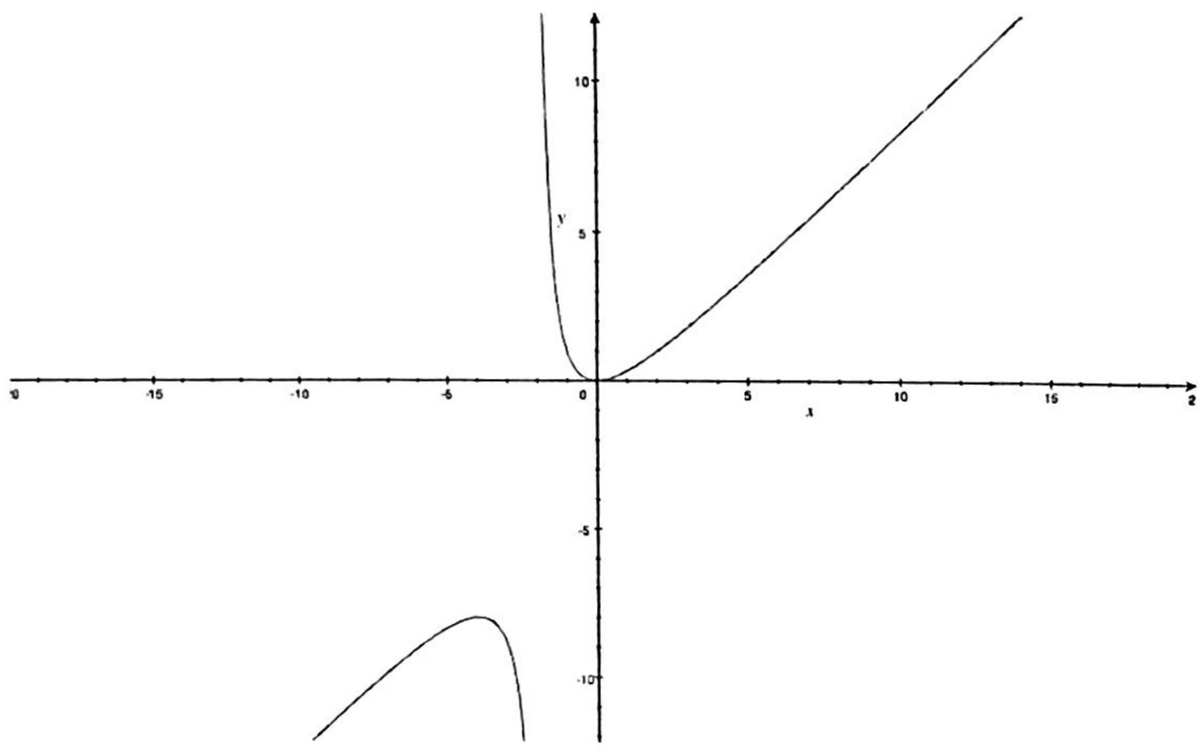
b) No symmetry

c) When x is large, y behaves like $y = x$ a straight line. When x is small, y behaves like $y = x^2/2$

d) Vertical asymptote at $x = -2$, if $x > -2$ then $y \to +\infty$, if $x < -2$ then $y \to -\infty$,

e) Minimum at $(0,0)$, maximum at $(-4,-8)$

$$y = \frac{x^2}{x+2}$$



vii) $y = x^2 + \frac{2}{x}$.

a) when $y = 0$ $x = -1.26$ so intercepts at $(-1.26,0)$

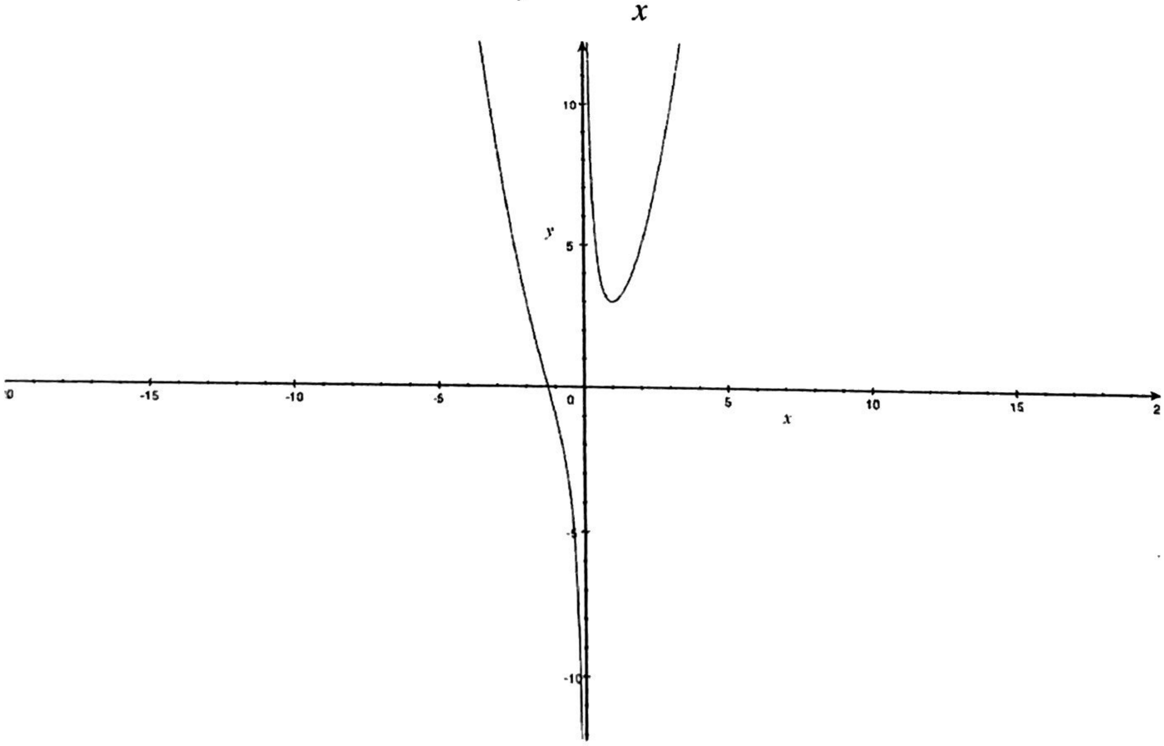
b) No symmetry

c) When x is large, y behaves like $y = x^2$ a straight line. When x is small, y behaves like $y = 2/x$

d) Vertical asymptote at $x = 0$, if $x > 0$ then $y \to +\infty$, if $x < 0$ then $y \to -\infty$,

e) Minimum at $(1,3)$, an inclined point of inflection at $(-1.26,0)$

$$y = x^2 + \frac{2}{x}$$



6. i) $y = \frac{\ln(x)}{x}$

$x > 0$

a) when $y = 0$ $x = e$ so intercepts at $(e,0)$

b) No symmetry

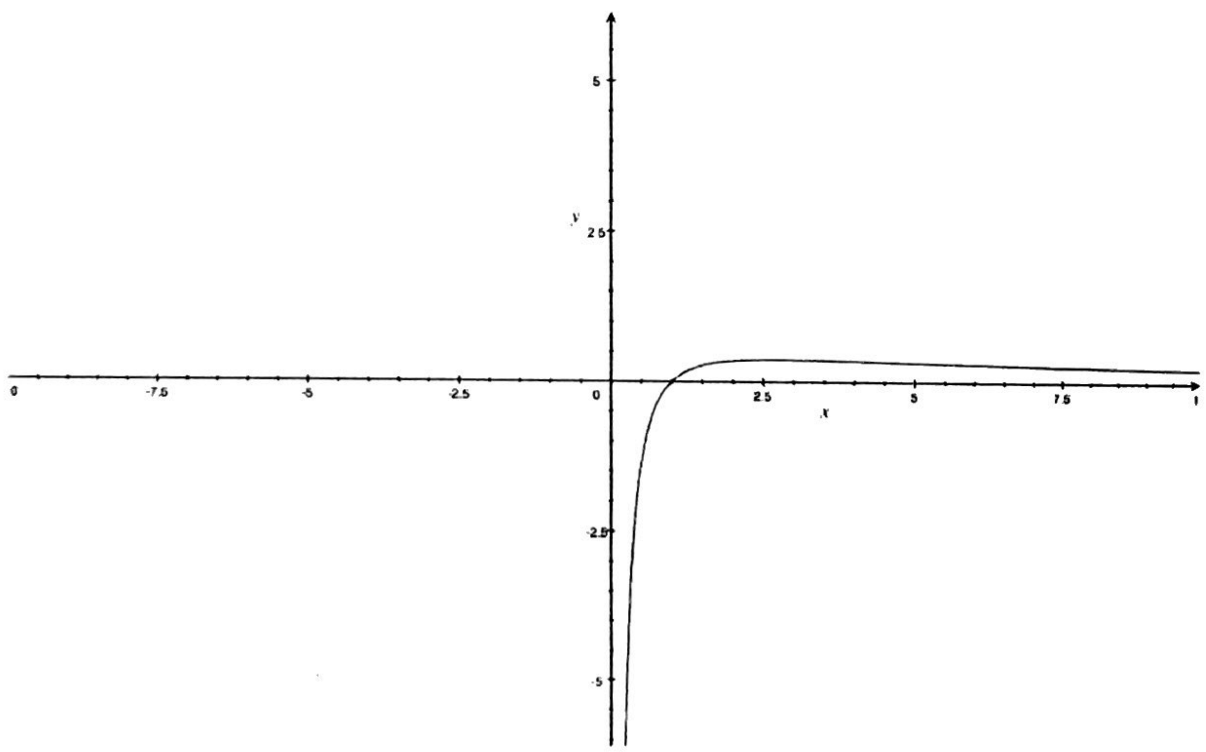
c) When x is large, y behaves like $y = x^2$ a straight line. When x is small, y behaves like $y = 2/x$

d) Horizontal asymptote at $y = 0$, $y \to 0$ as $x \to \infty$

Vertical asymptote at $x = 0$, $y \to -\infty$ as $x \to 0$

e) Maximum at $\left(e, \frac{1}{e}\right)$

$y = \frac{\ln(x)}{x}$



ii) $y = \frac{x}{1+x^2}$

a) when $x = 0$, $y = 0$ so intercept at (0,0)

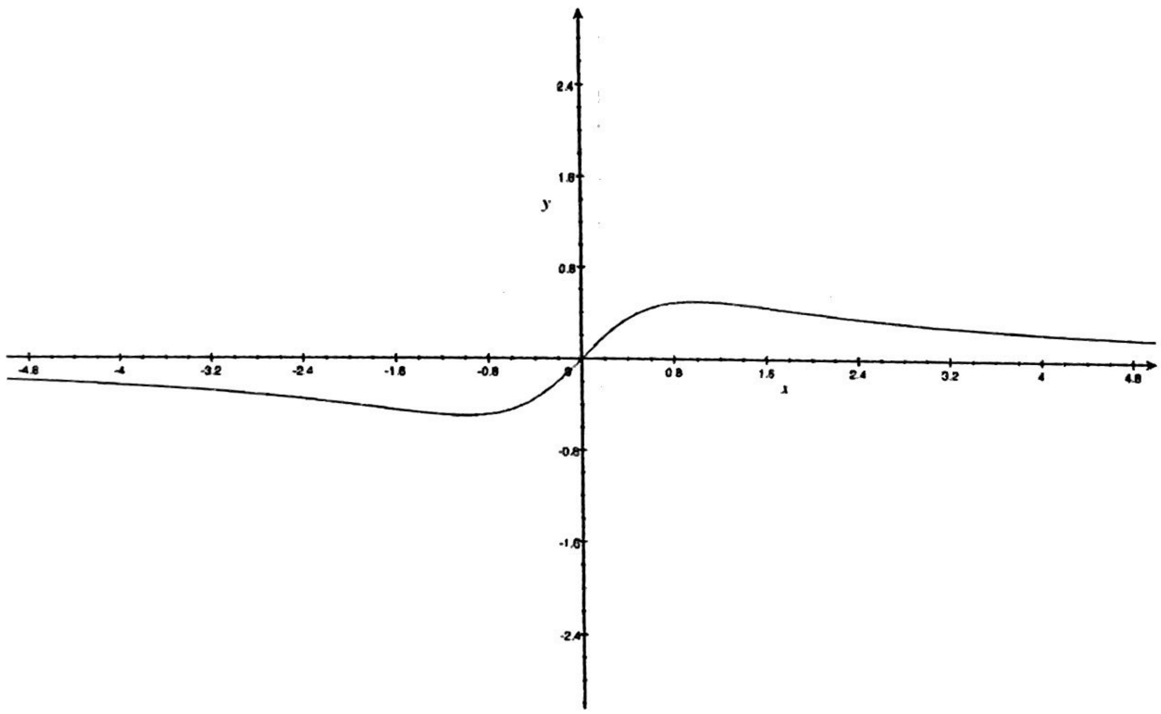
b) Rotational symmetry

c) When x is large, y behaves like $y = 0$ a straight line. When x is small, y behaves like $y = x$

d) Horizontal asymptote at $y = 0$, $y \to 0$ as $x \pm \to \infty$

e) Minimum at $\left(-1, -\frac{1}{2}\right)$, maximum at $\left(1, \frac{1}{2}\right)$

$$y = \frac{x}{1+x^2}$$



iii) $y = \dfrac{x^2 - 3x + 4}{x - 3}$

a) when $x = 0,\ y = -4/3$ so intercept at $(0, -4/3)$

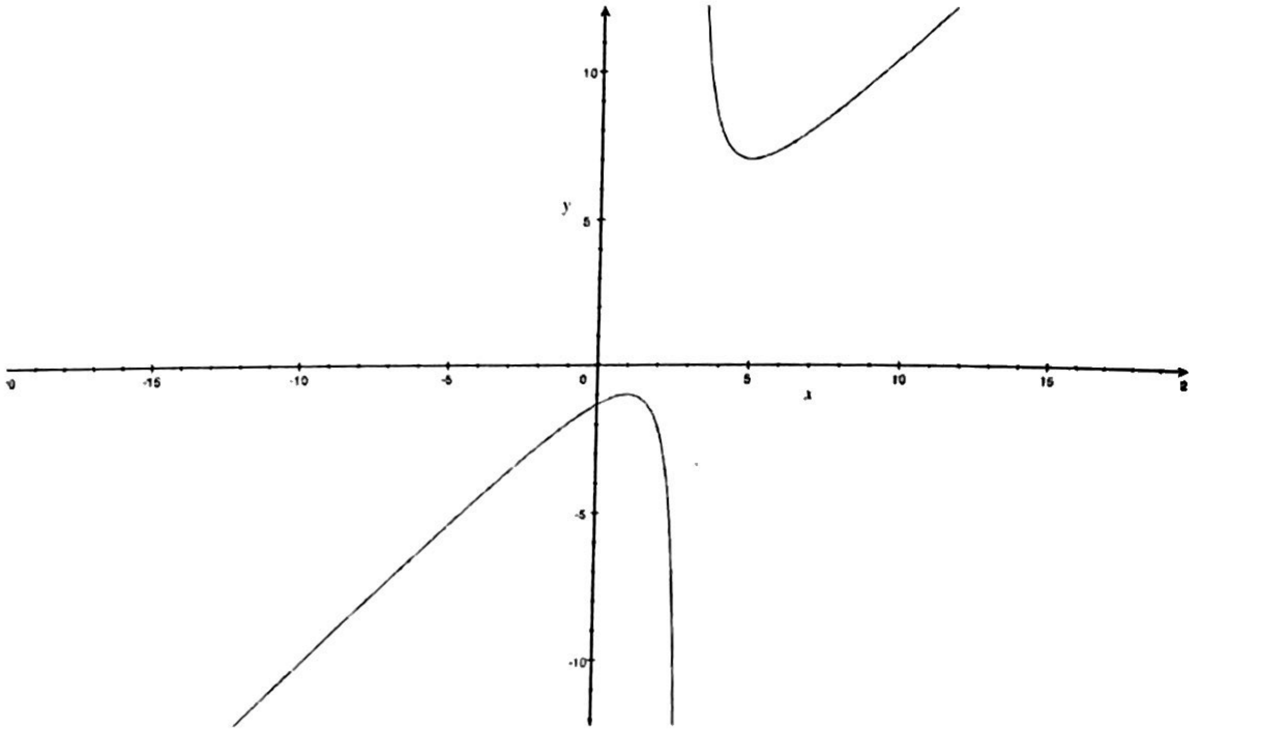
b) No symmetry

c) When x is large, y behaves like $y = x - 3$ a straight line. When x is small, y behaves like $y = 0$ $y = \dfrac{-4}{3} + \dfrac{5}{9}x$

d) Vertical asymptote at $x = 3,\ x > 3$ then $y \to +\infty,\ x < 3$ then $y \to -\infty$

e) Maximum at $(1, -1)$, minimum at $(5, 7)$

$$y = \frac{x^2 - 3x + 4}{x - 3}$$



iv) $y\left(x^2 - a^2\right) = x$

a) when $y = 0$ $x = 0$ so intercepts at (0,0)

b) Rotational symmetry

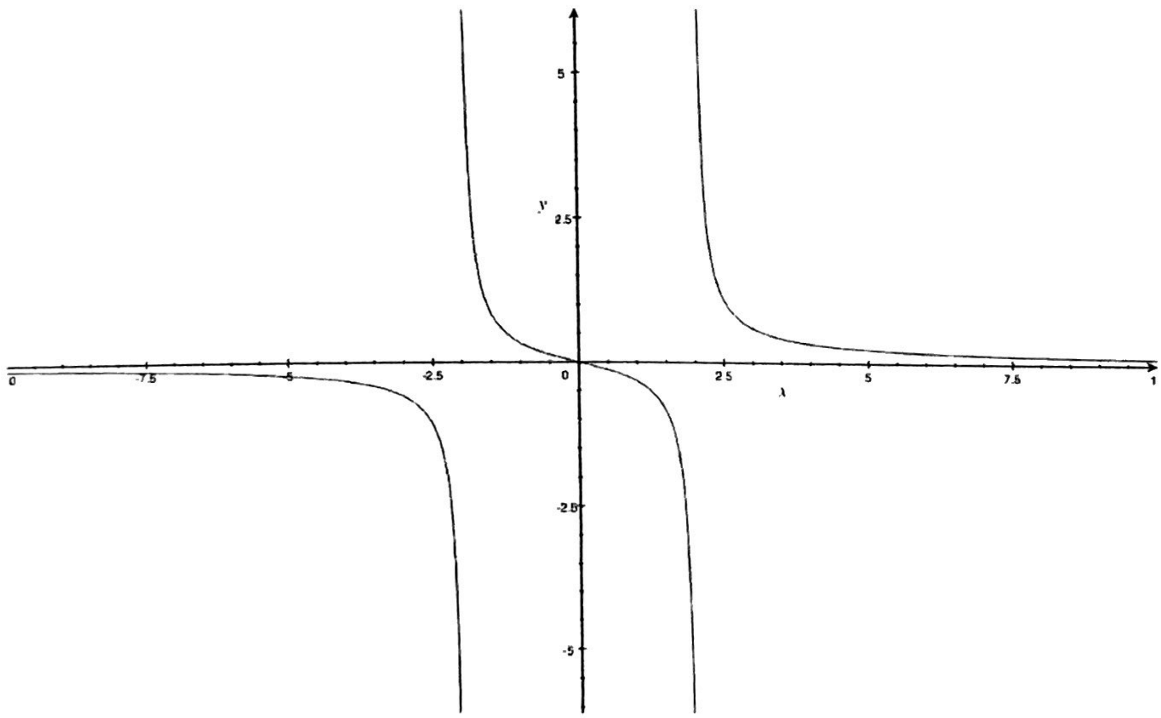
c) When x is large, y behaves like $y = 0$ a straight line. When x is small,

y behaves like $y = 2/x$

d) Vertical asymptote at $x = \pm a$, $y \to -\infty$ as $x \to 0$

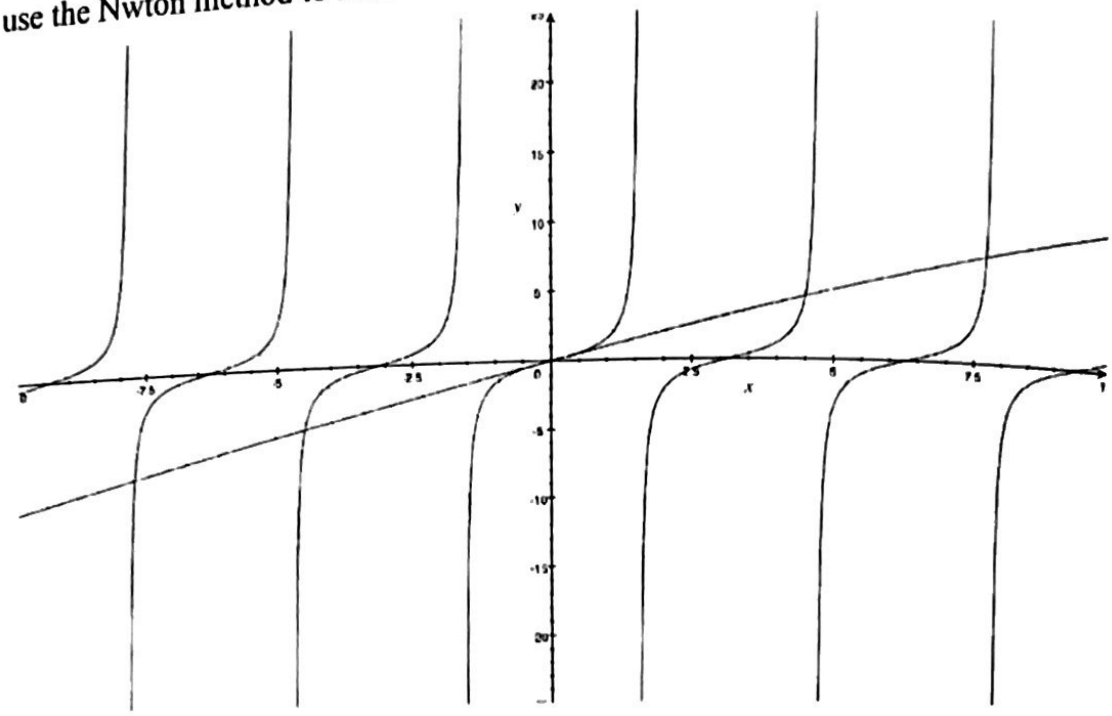
e) No turning points

$y\left(x^2 - a^2\right) = x$



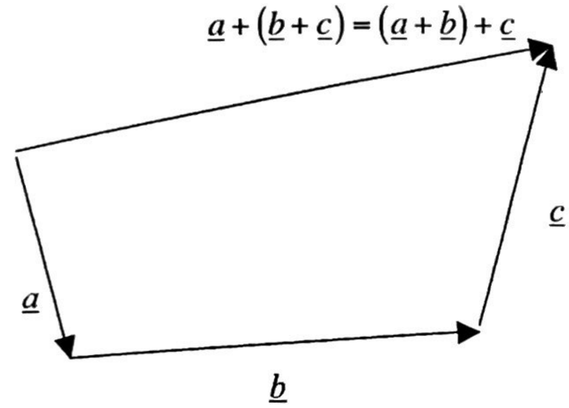
7. Sides are x and $\sqrt{\left(100 - x^2\right)}$. Find area and use the product rule to differentiate to find when maximum occurs.

8. This equation occurs when we consider the conduction of heat. Probably best to use the Nwton method to find $x = 4.4934$



Exercise 21.1 Solutions

1. Draw a four sided figure with sides $\underline{a}, \underline{b}$ and $\underline{c}$ and $\underline{a}+(\underline{b}+\underline{c})=(\underline{a}+\underline{b})+\underline{c}$



2. The vector $-\underline{r}$ has the same magniture as $\underline{r}$ but is in the opposite direction.

3. Draw a triangle with sides $\underline{a}$, $-\underline{b}$ and $\underline{c}$

4. Draw a triangle with sides $\underline{a}$, $\underline{b}$ and $\underline{a}+\underline{b}$.

5. Vectors in the same direction but with magnitude α times the magnitude of the original vector.

6. Three dimensional *xyz*-space drawing. Look up drawing of spherical polar coordinates

7. If $\underline{a}=-2\underline{i}+6\underline{j}$ and $\underline{b}=\underline{i}-\underline{j}$ then

   i) $|\underline{a}|=2\sqrt{10}$

   ii) $|\underline{b}|=\sqrt{2}$

   iii) $\underline{a}+\underline{b}=-\underline{i}+5\underline{j}, \quad |\underline{a}+\underline{b}|=\sqrt{26}$

   iv) $\underline{a}-\underline{b}=-3\underline{i}+7\underline{j}, \quad |\underline{a}-\underline{b}|=\sqrt{58}$

   vi) $3\underline{a}-2\underline{b}=-8\underline{i}+20\underline{j}, \quad |3\underline{a}-2\underline{b}|=4\sqrt{29}$

8. If $\underline{a} = (1,2,3)$ $\underline{b} = (3,6,9)$ then $3\underline{a} - \underline{b} = \underline{0} = (0,0,0)$.

9. Use the definitions and the cosine rule

10. i) Cauchy's Inequality. Let $\underline{v} = \alpha \underline{b} - \gamma \underline{a}$ where $\alpha = \underline{a} \bullet \underline{a}$, $\gamma = \underline{a} \bullet \underline{b}$ and consider $\underline{v} \bullet \underline{v}$

ii) Consider $\underline{a} + \underline{b}$ use Cauchy's inequality

iii) Consider $\underline{a} + \underline{b}$ use Cauchy's inequality

11. The product is equal to the product of two tangents. Try using the complex exponential form of the tangent function.

Exercise 22.1 Solutions

All from first principles. For example :-

$C = A + B \Rightarrow c_{ij} = a_{ij} + b_{ij}$ by definition of matrix addition

$= b_{ij} + a_{ij}$ since real numbers commute

$= B + A$ by definition of matrix addition

Hence $A + B = B + A$

Exercise 22.2 Solutions

1\. i) $\det\begin{pmatrix}1 & 1\\ 3 & 4\end{pmatrix}=1$ ii) $\det\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}=-2$

iii) $\det\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}=1$ iv) $\det\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}=1$

v) $\det\begin{pmatrix}3.1 & 2.6\\ -1.1 & 4.2\end{pmatrix}=15.88$ vi) $\det\begin{pmatrix}1 & 2\\ 3 & 8\end{pmatrix}=2$

vii) $\det\begin{pmatrix}1 & -2\\ 3 & -8\end{pmatrix}=-2$ viii) $\det\begin{pmatrix}-1 & 2\\ -3 & 8\end{pmatrix}=-2$

ix) $\det\begin{pmatrix}-1 & -2\\ 3 & 8\end{pmatrix}=-2$ x) $\det\begin{pmatrix}1 & 2\\ -3 & -8\end{pmatrix}=-2$

xi) $\det\begin{pmatrix}-1 & -2\\ -3 & -8\end{pmatrix}=2$ xii) $\det\begin{pmatrix}5 & 0\\ 4 & 3\end{pmatrix}=15$

xiii) $\det\begin{pmatrix}5 & 0\\ 0 & 3\end{pmatrix}=15$

xiv) $\det\begin{pmatrix}a & b\\ \lambda c & \lambda d\end{pmatrix}=\lambda(ad-bc)=\lambda\det\begin{pmatrix}a & b\\ c & d\end{pmatrix}$

2\. Evaluate the determinant of the following matrices

i) $\det\begin{pmatrix}1 & 2 & -1\\ 3 & 1 & 0\\ 2 & -1 & 2\end{pmatrix}=-5$ ii) $\det\begin{pmatrix}3 & 1 & 0\\ 1 & 2 & -1\\ 2 & -1 & 2\end{pmatrix}=5$

iii) $\det\begin{pmatrix}1 & 3 & -1\\ -1 & 1 & 4\\ 2 & 2 & 3\end{pmatrix}=32$ iv) $\det\begin{pmatrix}a & 2z & 9.6\\ 0 & b & \sqrt{2}\\ 0 & 0 & c\end{pmatrix}=abc$

v) $\det\begin{pmatrix}1 & 4 & 7\\ -1 & 3 & -2\\ 0 & 1 & 3\end{pmatrix}=16$ vi) $\det\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 4 & 9\end{pmatrix}=12$

vii) $\det\begin{pmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 3 & -1 & 7\end{pmatrix}=0$ viii) $\det\begin{pmatrix}a & b & c\\ -1 & x & 0\\ 0 & -1 & x\end{pmatrix}=ax^2+bx+c$

viii) $\det\begin{pmatrix}x & -1 & 0\\ 0 & x & -1\\ c & b & a+x\end{pmatrix}=x^3+ax^2+bx+c$

3. $\det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 27 \end{pmatrix} = 12$

4. To show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$ perform column operations. Look back at Question 2.

5. i) $A + B = \begin{pmatrix} 1 & 4 \\ 6 & 7 \end{pmatrix}$ $\quad A - B = \begin{pmatrix} 3 & -2 \\ -6 & 3 \end{pmatrix}$ $\quad C + D$ not possible

ii) $AB = \begin{pmatrix} 4 & 8 \\ 30 & 10 \end{pmatrix}$ $\quad BA = \begin{pmatrix} -2 & 14 \\ 12 & 16 \end{pmatrix}$ $\quad$ Note $AB \neq BA$

$AC = \begin{pmatrix} 4 & 10 & 16 & 22 \\ 10 & 20 & 30 & 40 \end{pmatrix}$ $\quad BC = \begin{pmatrix} 5 & 9 & 13 & 17 \\ 10 & 26 & 42 & 58 \end{pmatrix}$

CD, AE, and CE not possible

$DG = \begin{pmatrix} 8 & 10 & 18 \\ 9 & 5 & 11 \\ 3 & 6 & 8 \end{pmatrix}$ $\quad DF = \begin{pmatrix} -4 & 37 \\ -2 & 26 \\ -5 & 26 \end{pmatrix}$

$DE = \begin{pmatrix} 22 \\ 9 \\ 16 \end{pmatrix}$

6. Geometrically visualise the effect of the maps of the plane defined by the matrices

i) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has no effect. Matrix is the identity matrix

ii) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ reflection in $x = 0$ line

iii) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ reflection in $y = 0$ line

iv) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ stretch or expansion by a factor 2 in the x-direction

v) $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ stretch or expansion by a factor 3 in the y-direction

vi) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ reflection in $x = y$ line

vii) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ anti-clockwise rotation of $\pi/2$ about the origin

viii) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ clockwise rotation of $\pi/2$ about the origin

ix) $\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$ reflection in $x = 0$ line and an expansion by a factor of 2 in the x-direction

x) $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix}$ contraction by a factor of ½ in the x-direction and by 1/3 in the y-direction

xi) $\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ rotation about the origin plus expansion by a factor of 2 in both the x-direction and the y-direction

xii) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ Shear in x-direction increasing the size by a factor of 2

xii) $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ Shear in y-direction increasing the size by a factor of 3

7. $M_1 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ corresponds to a map that stretches the unit square by a factor 3 in the x-direction. $M_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ corresponds to a map that rotates the unit square by $\frac{\pi}{2}$ anti-clockwise. The composite map $M_1 M_2 = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$ rotates and then stretches while $M_2 M_1 = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix} \neq M_1 M_2$ since this stretches first before rotating. If you think of matrices as maps then there is no surprise that they do not commute.

8. A function is a linear map if we can write it as a matrix. In this case we can since if $z = x + iy$ then $f(z) = (a + ib)z = \begin{pmatrix} a & -b \\ ib & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

9. $AB = \begin{pmatrix} 3 & 6 \\ -1 & 4 \end{pmatrix}$ and $BA = \begin{pmatrix} 3 & 2 \\ -3 & 4 \end{pmatrix}$ and we see that $AB \neq BA$

10. $AB = \begin{pmatrix} 2 & 3 \\ 6 & 7 \end{pmatrix}$ $BA = \begin{pmatrix} 5 & 8 \\ 3 & 4 \end{pmatrix}$ AC not possible $CA = \begin{pmatrix} 4 & 8 \\ 5 & 8 \\ 0 & 2 \end{pmatrix}$

BC not possible $CB = \begin{pmatrix} 8 & 0 \\ 4 & 1 \\ 6 & -1 \end{pmatrix}$

11. First find $A^2 = A \times A$, then pre- or post- multiply by A.

12. $A^{-1} = \frac{1}{9}\begin{pmatrix} 2 & -7 \\ 1 & 1 \end{pmatrix}$ and $B^{-1} = \frac{1}{\alpha\beta}\begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}$ provided that $\alpha \neq 0$ and $\beta \neq 0$

Exercise 30.1 Solutions

1. Both Geometric series

i) $1+\frac{1}{10}+\frac{1}{100}+\ldots. = \frac{10}{9}$ ii) $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\ldots. = \frac{3}{4}$

2. $\sum_{k=1}^{n}\ln(k) = \ln(n!)$

3. The result we have to show has an integer involved and so .... proof by induction

4. $\frac{df(0)}{dx} = \frac{1}{2}$

5. $\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{\left(1-x^2\right)}}$

6. $y = \frac{1}{2}e^x\cos x + \frac{1}{2}e^x\sin x \Rightarrow \frac{dy}{dx} = e^x\cos x$

7. Use a Maclaurin series for $f(x) = e^x$

8. Use a binomial expansion

10. $f(x) = \frac{1}{1-x-2x^2} \approx 1 + x + 3x^2 + 5x^3$

11. Use $\ln\left(\frac{1}{2}\right) = \ln(1) - \ln(2)$

12. i) $y = x\ln(x) - x \Rightarrow \frac{dy}{dx} = \ln x$

ii) $y = xe^x - e^x \Rightarrow \frac{dy}{dx} = xe^x$

13. i) $\int \ln(x)dx = x\ln(x) - x$ ii) $\int x\ln(x)dx = \frac{x^2}{2}\ln(x) - \frac{x^2}{4}$

iii) $\int xe^x dx = xe^x - e^x$

14. $y = x^x \Rightarrow \frac{dy}{dx} = x^x\{1 + \ln(x)\}$

15. $n! \approx \frac{n^n}{e^{n+1}}$

16. i) $\int \frac{1}{t\ln(t)}dt = \ln(\ln(t)) + c$ ii) $\int_0^1 \frac{1}{\sqrt{(2+2t-t^2)}}dt = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right) = 0.62$

17. Starting with $\int_2^4 \frac{1}{t}dt$ use the change of variable $z = t/2$

18. i) $\frac{1}{4+5i} = \frac{1}{41}(4-5i)$ ii) $\frac{1-i}{2-i} = \frac{1}{5}(3-i)$

19. i) $\sqrt{(8i)} = 2+2i$ or $-2-2i$ ii) $\sqrt{(5-12i)} = -3+2i$ or $3-2i$

20. $z^2 - (1+2i)z - 1 + i = 0 \Rightarrow z = i,\ z = 1+i$ it is always worth checking that the answer satisfies the equation!

21. If $z = e^{2\pi i/5}$ then $z^5 = 1$ show that $z + z^4 = 2\cos\left(\frac{2\pi}{5}\right)$ hence result $\cos\left(\frac{2\pi}{5}\right) = 0.309$

22. Writing the surface area down as a function of radius and height, the best shape is when the ratio of radius to height is ½.

23. Can you show that c must be odd? (9,40,41) is the one with the biggest value of $c < 50$.

24. The answer is 10. One method to do this is looking in generating functions. One for later perhaps.